On the Use of Artificial Dissipation for Hyperbolic Problems and Multigrid Reduction in Time (MGRIT)

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1 Overview

We consider solving the linear advection equation in one spatial dimension with a parallel-in-time method. The model equation is

$$u_t - a u_x = f \tag{1}$$

on the domain $[0, 1.0] \times [0, t_f]$, with zero Dirichlet conditions in space. The initial condition in time is a sine-hump over the first half of the spatial domain (see Figure 1).

Solving even this simple problem scalably with a parallel-in-time method has so far proven elusive [4, 2].[1] We will focus on the multigrid reduction in time method (MGRIT) [1], which is equivalent to the earlier parareal method [3], in the specialized two-grid setting with F-relaxation. See [1] for a description of the method and of terms like F- and FCF-relaxation.

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[1]By "scalably", we mean an $O(N)$ method, where N is the number of unknowns, that converges in a fixed number of iterations to a fixed tolerance, regardless of problem size. Here, N is the number of time-steps.

Our motivation is the fact that greater dissipation in the problem leads to better MGRIT (and parareal) convergence [4]. Thus, we will explore different levels of artificial hyper-dissipation for this simple example. The discretization will always be consistent with the PDE in equation (1).

The discretization uses the second-order implicit SDIRK-2 method in time. In space, a standard centered fourth-order finite-differencing stencil is used. Specifically, let D^- and D^+ be the backwards and forwards first differencing matrices in space with stencils,

$$D^- := [-1 \;\; 1 \;\; 0], \tag{2}$$

$$D^+ := [0 \;\; 1 \;\; -1]. \tag{3}$$

And define $D^{(0)}$ with the centered stencil

$$D^{(0)} := [-1 \;\; 0 \;\; 1]. \tag{4}$$

Then our centered fourth-order first derivative operator is given by

$$D = \frac{1}{2h} D^{(0)} \left(I - \frac{1}{6} D^+ D^- \right), \text{ where} \tag{5}$$

$$Du \approx u_x + O(h^4), \tag{6}$$

and Du is a matrix vector product applied to the spatial vector u.

To complement that, we explore subtracting a fourth-order hyper-dissipation, i.e., a multiple of the fourth spatial derivative, from the spatial discretization. This fourth-order hyper-dissipation stencil is formed with

$$D^{(art)} = \frac{1}{h^4}(D^+ D^- D^+ D^-), \text{ where} \tag{7}$$

$$D^{(art)} u \approx u_{xxxx} + O(h^4). \tag{8}$$

The final spatial operator is then

$$(D - \gamma D^{(art)})u \approx u_x + O(\gamma) u_{xxxx}, \tag{9}$$

where γ is the amount of artificial hyper-dissipation added. For instance if $\gamma = h^{2.0}$, then we have a second order accurate spatial discretization with two orders of artificial hyper-dissipation. Traditional approaches to hyper-dissipation would typically consider $\gamma = h^{3.0}$, i.e., a third-order method in space. In this way, both D and $\gamma D^{(art)}$ would have a $1/h$ scaling.

2 Results

We now describe some preliminary runs on uniform space-time grids. Figure 1 depicts some sample solutions at various XBraid iterations for a $2^9 \times 2^7$ space-time grid. Figure 2 depicts solution profiles for the same problem size at various time-steps. The higher-order accuracy is already visible, as the solution has not noticeable decayed.

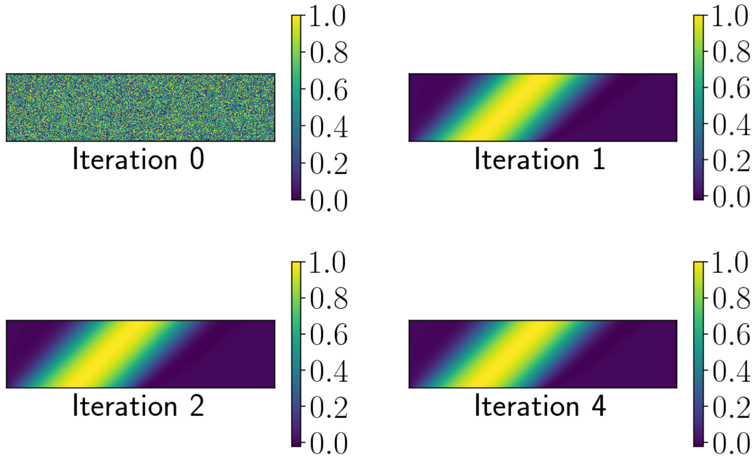

Figure 1: Solution snapshots during XBraid iterations.

The test setup is as follows.

- Initial condition in space is a sine hump over the first half of the domain, and zero everywhere else.
- Random initial space-time guess. Thus, our experiments measure the asymptotic convergence rate regardless of the initial condition.
- Coarsen to 2 points in time, yielding a truly multilevel solver
- FCF-relaxation
- Residual stopping tolerance of $\frac{10^{-8}}{\sqrt{\delta t\, h}}$
- CFL number of $1.0 = \frac{\delta t}{h}$, implying $t_f = 0.25$.
- No spatial coarsening
- SDIRK-2 in time and the spatial discretization from equation (9)
- Experiments will explore: temporal coarsening factors of $m = 2$ and $m = 4$, V- and F-cycles, and different orders of hyper-dissipation γ.

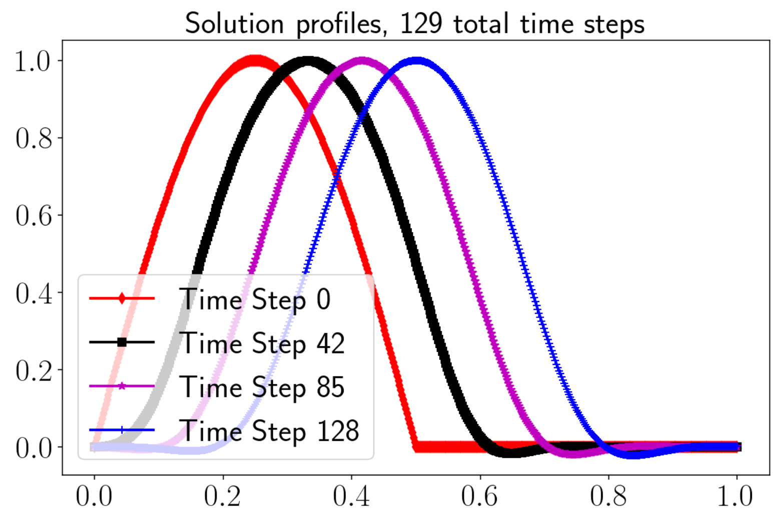

Figure 2: Solution profiles during XBraid iterations.

The results are depicted in Table 1. The use of "large" hyper-dissipation at $\gamma = h^{2.0}$ leads to very fast and scalable F-cycle iteration counts. Remember, despite the hyper-dissipation, this is a second order method in space and time. Moreover, this approach appears to also work for the larger coarsening factors in time of $m = 4$. This should be compared to the more traditional third-order in space discretization of $\gamma = h^{3.0}$, where the iteration counts grow uncontrollably. The $\gamma = h^{2.5}$ discretization (spatial order 2.5 discretization) is somewhere in between, more scalable, but not completely so.

γ	Cyc	m	Grid: $2^9 \times 2^7$	$2^{10} \times 2^8$	$2^{11} \times 2^9$	$2^{12} \times 2^{10}$	$2^{13} \times 2^{11}$
$h^{3.0}$	F	2	6	8	11	18	31
$h^{2.5}$	V	2	5	7	10	14	22
$h^{2.5}$	F	2	4	4	4	4	5
$h^{2.5}$	F	4	4	5	5	6	9
$h^{2.0}$	V	2	4	4	5	6	7
$h^{2.0}$	F	2	4	4	4	3	3
$h^{2.0}$	F	4	4	3	3	3	3

Table 1: MGRIT Iteration counts for various values of γ, cycle type and temporal coarsening factor m.

3 Conclusions

This study indicates that there are trade-offs between order of accuracy in space and speed of convergence for MGRIT. In fact, one can achieve previously unattainable convergence rates for advection, if willing to sacrifice an order of accuracy in space.

Regarding the generality of this result, experiments were run with classic first-order upwinding in space, where the spatial discretization is defined with the operator from equation (2), and backward Euler in time. If the order of the spatial discretization is decreased from $O(h)$ to $O(h^{2/3})$, through the addition of extra artificial dissipation, then similarly fast F-cycle results are observed.

The equivalence between MGRIT and parareal for the special two-grid with F-relaxation case implies that some of these techniques could be used for parareal.

Lastly, there are downsides to this approach. One should use an implicit scheme, because the numerical stability for explicit time-stepping is now determined by the artificial hyper-dissipation term. And to use an implicit scheme, one must invert spatial matrices representing higher derivatives. These downsides are in addition to losing accuracy in space, although overall higher-order spatial discretizations can obviously be maintained.

References

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