Advening to adynkrafields: Young tableaux to component fields of the 10D, $\mathcal{N} = 1$ scalar superfield

S. James Gates, Jr., Yangrui Hu, and S.-N. Hazel Mak

Starting from higher dimensional adinkras constructed with nodes referenced by Dynkin Labels, we define "adynkras." These suggest a computationally direct way to describe the component fields contained within supermultiplets in all superspaces. We explicitly discuss the cases of ten dimensional superspaces. We show this is possible by replacing conventional θ-expansions by expansions over Young Tableaux and component fields by Dynkin Labels. Without the need to introduce σ-matrices, this permits rapid passages from Adynkras → Young Tableaux → Component Field Index Structures for both bosonic and fermionic fields while increasing computational efficiency compared to the starting point that uses superfields. In order to reach our goal, this work introduces a new graphical method, "tying rules," that provides an alternative to Littlewood's 1950 mathematical results which proved branching rules result from using a specific Schur function series. The ultimate point of this line of reasoning is the introduction of mathematical expansions based on Young Tableaux and that are algorithmically superior to superfields. The expansions are given the name of "adynkrafields" as they combine the concepts of adinkras and Dynkin Labels.

1. Introduction

Papert [1] introduced the phrase "computational thinking" in 1980. A search on-line can be found to lead to the following comment.

> *Computational Thinking (CT)... is essential to the development of computer applications, but it can also be used to support problem solving across all disciplines, including math, science, and the humanities.*

One point this approach emphasizes is focused attention on the formulation of algorithms. In recent works [2, 3], we have been exploring emerging opportunities created by the adinkra-based framework, enhanced algorithmic architectures, and computational applications to study superspace[1] supergravity in the ten and eleven dimensional geometrical limits of the heterotic string, superstrings, and M-Theory. These efforts have shown success as they permitted the *complete* deciphering of the Lorentz spectra in $\mathfrak{so}(1, 10)$ and $\mathfrak{so}(1, 9)$, respectively, for all component fields contained in scalar superfields.

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[1] It is an often overlooked historical fact that the concept of "superspace" [4] was introduced independently and separately from the concept of "superfields"[5] and we recognize S. Kuzenko for discussion.

We posit this as a notable advance against the benchmark established by Bergshoeff and de Roo [6] since our results cover Type-II superspaces and 11D superspace.

Adinkras appropriate for 10D and 11D superfields, involving billions of degrees of freedom, have been successfully constructed by use of Young Tableaux [3], Dynkin Labels [7–9], and Plethysm [8, 10, 11]. Successful algorithms based on the information held *solely* in these adinkras, as opposed to that in traditional θ-expansions of superfields, thus emerged due to their increased calculational and computational efficiency.

The efficacy of this approach can be understood if we analogize a superfield to a biological body where the adinkra plays the role of a genome. By the study of genes and knowing their expressions, one can deduce information about structures. This is the reason why using the foundation of the adinkra concept, we were able to analyze [3] *all* the 2^{31} ($= 2,147,483,648$) bosonic degrees of freedom and *all* the 2^{31} ($= 2,147,483,648$) fermionic degrees of freedom in the 11D, $\mathcal{N} = 1$ scalar superfield. Scalar superfields act as gateways to the similar deciphering the component field spectra of superfields in all spin representations. Using this fact, we have begun the task of identifying superfields that contain the conformal graviton in these contexts.

By this means we discovered, a bit surprisingly, the 11D, $\mathcal{N} = 1$ scalar superfield contains:

(a.) the symmetrical conformal graviton at the 16-th order of the θ-expansion,

(b.) a 3-form at the 16-th order of the θ-expansion,

(c.) a conformal gravitino at the 17-th order of the θ-expansion, and

(d.) 1,494 bosonic fields and 1,186 fermionic fields in general,

... facts unknown from the time this theory was introduced into the literature.

Furthermore, we found the 11D, $\mathcal{N} = 1$ scalar superfield does not possess the antisymmetrical part of the component level vielbein at the sixteenth level. Based on past experience with supergravity in superspace [12], combined with the results from the study of the 11D, $\mathcal{N} = 1$ scalar superfield, the simplest proposal for the 11D, $\mathcal{N} = 1$ supergravity prepotential is a spinor superfield $\Psi_\alpha{}^2$ where the complete component fields of Poincaré vielbein are contained at the 17-th level, along with the gauge 3-form, and the

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[2] Here, Ψ_α is an 11D, $\mathcal{N} = 1$ superfield, not to be confused with the 10D, $\mathcal{N} = 1$ superfield that we denote by the same symbol later in this work.

complete component fields of Poincaré gravitino are contained at the 18-th level in the θ-expansion.

Another surprise uncovered was the prolific presence of the component graviton among 11D, $\mathcal{N} = 1$ candidates for the SG prepotential. Our scan reveals this particular presence of supergravity on-shell component occurs in every superfield up to and including the $\Psi_{\{255255\}} = \Psi_{[2,0,2,0,0]} = \Psi_{\{\underline{a}_3,\underline{a}_4|\underline{a}_1\underline{b}_1\underline{c}_1,\underline{a}_2\underline{b}_2\underline{c}_2\}}^{3}$.

It may not be obvious if one begins with higher dimensional adinkras described by Dynkin Labels, that there is a path to component fields. It is the purpose of this work to provide an end-to-end demonstration showing how this is carried out. We will marshal the lessons learned in the 11D, $\mathcal{N} = 1$ theory and apply them to the 10D, $\mathcal{N} = 1$ scalar superfield as it is the basis for gaining a complete understanding of off-shell supersymmetical theories in this arena.

In this work, we will introduce a concept which shows some potential for becoming a computationally superior complement for superfields. We call it the "adynkrafield formulation" which appears as a natural consequence of the path we have explored. The importance of two distinct sets of Young Tableaux, each associated with Dynkin Labels in our discussions, points toward the use of the Young Tableaux together with the introduction of a "level parameter" ℓ as a basis for expansions. Initial evidence is given that these are sufficient to accurately investigate the domain where heretofore traditional Grassmann coordinates in superfields, as defined by Salam and Strathdee, provided the sole means enabling investigations.

In Chapter two, we present the adinkra for the 10D, $\mathcal{N} = 1$ scalar superfield that provides the starting point for our construction. All nodes of the adinkra are described by Dynkin Labels. We review how computational efficiency is gained from this view point. Since adinkras utilizing Dynkin Labels play a key role, the new term "adynkras" is introduced to describe this particular form of adinkras.

In Chapter three, we turn to adapting the well known technology of Young Tableaux to the task of representing the irreducible bosonic representations of $\mathfrak{so}(10)$. It should be noted the challenge here comes about because there is a well accepted method for using Young Tableaux to represent the irreducible representations for $\mathfrak{su}(10)$, but not for $\mathfrak{so}(10)$. Adaptations

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[3] This translation of notations between dimensions, Dynkin labels and the index notation of an irrep will be explicitly explained in the following sections.

are necessary to allow a set of decorated Young Tableaux to accomplish the latter goal.

In the literature, a number of other works have offered proposals for such adaptations to achieve this purpose [13–17]. Several of them use a very similar approach to ours [13–16], which translates Dynkin labels to Young Tableaux, and puts bosonic and spinorial Young Tableaux side by side. Others associate irreducible representations with skew Young Tableaux which involves "negative boxes" for tensor product calculations [17].

Our discussion starts with the presentation of a logical path for the construction of a projection matrix for $\mathfrak{su}(10) \supset \mathfrak{so}(10)$. Here we set our conventions for the embedding. We next adopt a set of conventions for how the Dynkin Labels are to be translated into conventional $\mathfrak{so}(10)$ Young Tableaux. This is followed by a brief discussion of how these $\mathfrak{so}(10)$ Young Tableaux are reducible with respect to a set of "adapted" $\mathfrak{so}(10)$ Young Tableaux we introduce in this work and the extraction of irreducible tableaux.

Chapter four is devoted to the graphical rules for the $\mathfrak{su}(10) \supset \mathfrak{so}(10)$ branching rules. Littlewood's rule is reviewed. Then we turn to the introduction of a set of graphical rules, we call "tying rules," that allow the $\mathfrak{su}(10)$ Young Tableaux to be decorated in such a way so as to produce irreducible $\mathfrak{so}(10)$ Young Tableaux. This is illustrated in some examples. However, this discussion is limited to bosonic Young Tableaux, i.e. those Tableaux that are associated with bosonic representations.

Chapter five aims to construct Young Tableau representations for the spinorial irreducible representations of $\mathfrak{so}(10)$. Mixed Young Tableaux as well as the graphical rules that lead to correct dimensions are introduced. Mixed Young Tableaux highlight the facts that we are using two distinct types of Young Tableaux. Blue Tableaux are associated with bosonic indices and representations while red Tableaux are associated with spinor indices and representations. This is accompanied by the concomitant task of describing a corresponding set of Dynkin Labels. Illustrations are given to show how the issue of irreducibility is handled.

Chapter six presents the general graphical rules to get tensor product decompositions of a bosonic irrep with the basic spinor representation of $\mathfrak{so}(10)$. The inverse relation between this tensor product rule and the dimension rule in last chapter is presented and demonstrated through examples.

Chapter seven brings all the strands of the previous chapters together with the explicit presentation of the field variables showing all their various types of indices associated with each node of the adynkra introduced in Chapter two. The derivation begins from the adynkra in chapter two in which the nodes are expressed in terms of Dynkin Labels. It is shown the

Dynkin Labels contain sufficient data to derived a complete description of the Lorentz structure of all the component fields. A complete description of the irreducibility conditions is presented.

Chapter eight presents a new concept to which we give the name "adynkrafields." Adinkras and adynkras do not necessarily depend on field variables. In such settings, the adinkras and adynkras play a role similar to matrices in, for example, the study of $\mathfrak{su}(3)$ representations.

Chapter nine presents our conclusions.

There are three appendices included in this work. Appendix A is devoted to present a dictionary between bosonic indices of field variables, irreducible Young Tableaux, and Dynkin Labels. Appendix B gives explicit examples of the dimension formula for Mixed Young Tableaux defined in Chapter five. Appendix C is devoted to explicit examples of the graphical tensor product rules stated in Chapter six as well as providing demonstrations of the subject of multiplication of the fundamental SYT by BYT's to obtain SYT's. It thus covers the same topics as Chapter 5 but now translates spinorial representations into field language by only considering tensor product decomposition.

The final portion of this paper includes our references.

2. The 10D, $\mathcal{N} = 1$ scalar superfield adynkra

In the work of [3] the adinkra for the 10D, $\mathcal{N} = 1$ scalar superfield, with nodes expressed in terms of Dynkin Labels, was shown as it appears in Figure 1. Counting the number of open and closed nodes respectively implies this is a superfield with 15 bosonic component fields and 12 fermionic component fields. The Dynkin Labels on each node carry the information about the SO(1,9) Lorentz representation of each field. However, the image signifies another possibility to which we will return shortly.

In most of our initial investigations of higher dimensional adinkras, the bulk of the discussions was carried out in cases where the dimensionality of the representations of the nodes was illustrated. There is an ambiguity in such labeling. We can see this by considering the dimensionality formula in the simpler case of $\mathfrak{su}(3)$. In the next chapter shown in equation (3.2), there appears the relation between the dimensionality of a representation in $\mathfrak{su}(3)$ specified by the integers p and q which occur in a Dynkin Label $[p, q]$. The form of $d(p, q)$ shows that the case where $p = $ M and $q = $ N has the same dimensionality as the case where $p = $ N and $q = $ M for any positive integers M and N. So for a fixed value of $d(p, q)$, the solutions for p and q
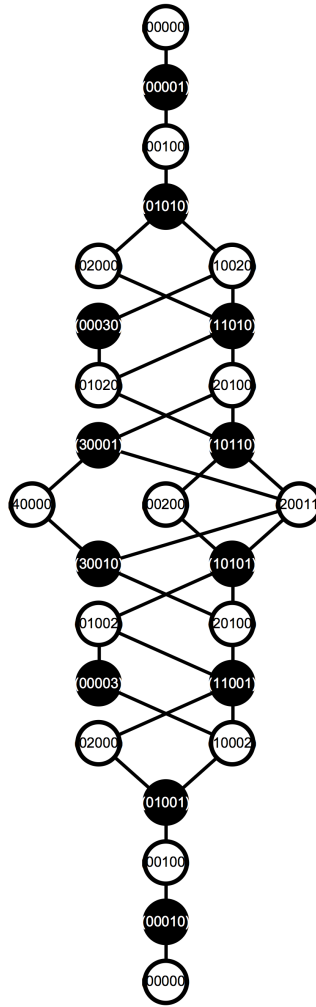
Figure 1: Adinkra Diagram for 10D, $\mathcal{N} = 1$ Scalar Superfield

are not unique. Thus, labeling the nodes of the higher dimensional adinkra with Dynkin Labels removes the ambiguity.

The image in Figure 1 uses Dynkin Label to describe the Lorentz representation associated with each node. In the following, it will be shown that this realization has a computationally superior property. The knowledge of the integers that appear in the Dynkin Label suffices to derive complete component level field variables with their varied Lorentz index structures. Henceforth, we will refer to adinkras where their nodes are described by Dynkin Labels as "adynkras," replacing the letters "ink" in "adinkra" by the letters "ynk" from "Dynkin."

The "platform" of this current work is within the context of linearized Nordström supergravity in 10D, $\mathcal{N} = 1$ superspace, as we established the foundation for this in the work of [18]. One reason for performing this is the component level construction of the linearized 10D, $\mathcal{N} = 1$ Nordström supergravity theory yields the simplest context in which the derivation faces all the same problems present in the general class of models described previously [2, 3, 18].

In order to accomplish our task we need to develop:

(a.) a set of new concepts based on direct graphical manipulations of Young Tableaux allowing them to generate $\mathfrak{su}(10) \supset \mathfrak{so}(10)$ branching rules, and

(b.) a "translation dictionary" for Dynkin Labels into indices on field variables.

In particular for task (a.), to our knowledge, these will be new concepts and techniques introduced into the literature.

We need a well defined methodology for converting Dynkin Labels into indices on a set of field variables. We now turn to this in the less complicated context of the 10D, $\mathcal{N} = 1$ system [18].

Almost since their introduction [19] and in one form or another, numbers of physicists have posed the question, "Is the purpose of adinkras to replace superfields?" We have always responded, "No, the purpose of adinkras is to augment Salam-Strathdee superfields" [5].

From the time of the discovery of supersymmetry at the level of on-shell component representations [20–22], the description's incompleteness was obvious. To remedy this, Salam and Strathdee invented superfields. However, superfields are ill-posed as a convenient platform from which to study the structure of the representation theory associated with spacetime supersymmetry. A most forceful demonstration of this was given in our work of [3], where we demonstrated the unwieldiness associated with the θ-expansion

of the scalar 11D, $\mathcal{N} = 1$ superfield itself acts as a computational impediment to deriving results. The problem arises due to the fact that the *actual derivation* of component results require the subsidiary derivation of large numbers of Fierz identities. In the context of this system, the totality of these derivations has never been shown in the literature. We suspect the reason is the daunting numbers of these.

The primary purpose of adinkras is to provide a "Goldilocks" solution by avoiding the incompleteness of the component-level approach while simultaneously avoiding the unwieldiness of the superfield approach. This is accomplished by banishing the need for γ-matrices (actually σ-matrices in 10D) in deriving supersymmetrical representation theory results.

3. Bosonic irreps of $\mathfrak{so}(10)$ & irreducible bosonic young tableaux

3.1. Preview: $\mathfrak{su}(3)$ & irreducible bosonic young tableaux

In the manner of a warm-up, we review a discussion from a previous paper [23] regarding the $\mathfrak{su}(3)$ algebra. As covered there, we have seen the most general $\mathfrak{su}(3)$ Young Tableaux takes the form

(3.1)

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We next introduce the Dynkin Label $[p, q]$ as the highest weight vector for irreps in $\mathfrak{su}(3)$. The dimensionality of the irrep with the Dynkin Label $[p, q]$ is given by the Weyl dimension formula applied to $\mathfrak{su}(3)$ [24]

(3.2) $\qquad d(p, q) = \frac{1}{2} (p + 1) (q + 1) (p + q + 2) \quad .$

3.2. Feature: beginning $\mathfrak{su}(10) \to \mathfrak{so}(10)$ & irreducible bosonic young tableaux

We now look at the bosonic Young Tableaux that correspond to bosonic representations of $\mathfrak{so}(10)$. Generally speaking, these are not necessarily irreducible. In order to establish a graphical language to describe bosonic irreducible representations in $\mathfrak{so}(10)$, we must define *irreducible* bosonic Young Tableaux.

Consider the projection matrix for $\mathfrak{su}(10) \supset \mathfrak{so}(10)$ [7],

$$(3.3) \qquad P_{\mathfrak{su}(10)\supset\mathfrak{so}(10)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \end{pmatrix} \quad .$$

The highest weight of a specified irrep in $\mathfrak{su}(10)$ is a row vector $[p, q, r, s, t, u, v, w, x]$, where p to x are non-negative integers. Since the $\mathfrak{su}(10)$ YT with n vertical boxes is the conjugate of the one with $10 - n$ vertical boxes, we need only consider the $u = v = w = x = 0$ case.

Starting from the weight vector $[p, q, r, s, t, 0, 0, 0, 0]$ in $\mathfrak{su}(10)$, we define its projected weight vector $[p, q, r, s, s + 2t]$ in $\mathfrak{so}(10)$ as the Dynkin Label of the corresponding irreducible bosonic Young Tableau.

$$(3.4) \qquad [p, q, r, s, s + 2t] = [p, q, r, s, t, 0, 0, 0, 0] \, P^{T}_{\mathfrak{su}(10)\supset\mathfrak{so}(10)} \quad .$$

Thus, given an irreducible bosonic Young Tableau, we obtain the definition of its corresponding bosonic irreducible Dynkin Label representation. Moreover, we can reverse this process and show the one-to-one correspondence between bosonic Dynkin Label irreps and irreducible bosonic Young Tableaux. Namely, given a Dynkin Label $[a, b, c, d, e]$, write a set of linear equations:

$$(3.5) \qquad \begin{aligned} a &= p \quad , \\ b &= q \quad , \\ c &= r \quad , \\ d &= s \quad , \\ e &= s + 2t \quad , \end{aligned}$$

and obtain

$$(3.6) \qquad \begin{aligned} p &= a \quad , \\ q &= b \quad , \\ r &= c \quad , \\ s &= d \quad , \\ t &= \frac{e - d}{2} \quad . \end{aligned}$$

The problem is that in order to obtain a valid YT, p, q, r, s, and t have to be non-negative integers, meaning $e - d$ has to be an non-negative even integer. When $e > d$, we need to prove that $e - d$ is even and assign the corresponding YT with subscript $(IR, +)$. When $e < d$, we assign the corresponding YT with subscript $(IR, -)$ which is described by the alternate Dynkin Label given by $[a, b, c, e, d]$. In this case, $p = a$, $q = b$, $r = c$, $s = e$, and $t = \frac{d-e}{2}$, where $d - e$ has to be even.

The proof is simple. Consider the congruence classes of a representation with Dynkin Label $[a, b, c, d, e]$ in SO(10),

$$(3.7) \qquad \begin{aligned} C_{c1}(R) &:= d + e \pmod 2 \quad , \\ C_{c2}(R) &:= 2a + 2c + 3d + 5e \pmod 4 \quad . \end{aligned}$$

A congruence class is an equivalent class of irreps. Based on the above equations, there are totally four congruence classes in SO(10),

$$(3.8) \qquad \begin{bmatrix} C_{c1}, C_{c2} \end{bmatrix}(R) = \begin{cases} [0, 0] \\ [0, 2] \\ [1, 1] \\ [1, 3] \end{cases} \quad .$$

The quantity $\begin{bmatrix} C_{c1}, C_{c2} \end{bmatrix}$ can be treated as a vector and it satisfies

$$(3.9) \quad \begin{bmatrix} C_{c1}, C_{c2} \end{bmatrix}(R_1 \otimes R_2)$$
$$= \begin{bmatrix} C_{c1}(R_1) + C_{c1}(R_2) \pmod 2 , \ C_{c2}(R_1) + C_{c2}(R_2) \pmod 4 \end{bmatrix} \quad .$$

We know that

$$(3.10) \qquad \begin{cases} \text{bosonic irrep} \otimes \text{bosonic irrep} = \text{bosonic irrep} \quad , \\ \text{bosonic irrep} \otimes \text{spinorial irrep} = \text{spinorial irrep} \quad , \\ \text{spinorial irrep} \otimes \text{spinorial irrep} = \text{bosonic irrep} \quad . \end{cases}$$

One can quickly check that $C_{c1}(R)$ actually classifies the bosonic irreps and spinorial irreps: $C_{c1}(R) = 0$ is bosonic and $C_{c1}(R) = 1$ is spinorial. Consequently, a bosonic irrep satisfies $d + e = 0 \pmod 2$ and consequently $d - e = 0 \pmod 2$.

Summarizing, given an irreducible bosonic Young Tableau with p columns of one box, q columns of two vertical boxes, r columns of three vertical boxes, s columns of four vertical boxes, and t columns of five vertical

boxes, the Dynkin Label of its corresponding bosonic irrep is $[p, q, r, s, s + 2t]$ or $[p, q, r, s + 2t, s]$ depending on its self-duality. Given a bosonic irrep with Dynkin Label $[a, b, c, d, e]$, its corresponding irreducible bosonic Young Tableau is composed of a columns of one box, b columns of two vertical boxes, c columns of three vertical boxes, d columns of four vertical boxes, and $|e - d|/2$ columns of five vertical boxes. Duality properties depend on the sign of $e - d$ which has been discussed. From the discussion above, we know that a bijection is established between Dynkin Labels and BYTs, that there's no ambiguity in the translation from Dynkin Labels to BYTs. When $d = e$ (even), it would mean d columns of four vertical boxes, instead of sticking two sets of $e/2$ columns of five boxes of opposite dualities.

The simplest examples, also the fundamental building blocks of a BYT, are given below.

(3.11)

We put "IR" as subscripts to indicate that these are irreducible representations[4]. Putting together these columns corresponds to adding their Dynkin Labels. All the BYT with one or more columns of 5 boxes can be either self-dual or anti-self-dual. Here we impose a rule that if there's no + or − subscript put at the corner, it is assumed as the direct sum of the two irreps.

(3.12)

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[4] This convention is also adopted by [14].

With these basic elements, we can build different examples of BYTs,

$$(3.13) \qquad \equiv [2, 0, 1, 0, 0] \quad,$$

$$(3.14) \qquad \equiv [1, 0, 0, 1, 3] \quad,$$

$$(3.15) \qquad \equiv [1, 0, 0, 3, 1] \quad.$$

These last two images illustrate the meaning of "duality" in the present context. The Tableaux shown in (3.14) and (3.15) each corresponds to tensors that possess ten indices, five of which are totally antisymmetric as signified by the length of the first column. Thus, with respect to these indices the tensor are five-forms. It is a well recognized fact that a five-form in the context of a ten dimensional manifold can either be dual or anti-dual. This distinction is captured by the $\pm$ subscript shown at the bottom of the tableaux.

3.3. Indices corresponding to irreducible bosonic young tableaux

When translating the irreducible bosonic Young Tableaux into field representations, Young Tableaux tell us the index structure of the field. In some literature [25, 26], for the efficiency in expressing an index structure, an entire Young Tableau is drawn in the subscript of the field in replacement of a bunch of overlapping ( ) and [ ]. Here we develop the notation further such that it becomes compact and typable. We introduce the following notational conventions. We put all the bosonic indices in a pair of curly braces "{ }". We use "|" to separate indices in column(s) of YT with different heights and "," to separate indices in column(s) of YT with the same heights. It should be noted that the { }-indices, irreducible bosonic Young Tableaux, and Dynkin Labels are equivalent and have one-to-one correspondence. The general expression is given below in Figure 2 and Figure 3,

$$\{\underline{a}_1, \ldots, \underline{a}_p | \underline{b}_1 \underline{c}_1, \ldots, \underline{b}_q \underline{c}_q | \underline{d}_1 \underline{e}_1 \underline{f}_1, \ldots, \underline{d}_r \underline{e}_r \underline{f}_r | \underline{g}_1 \underline{h}_1 \underline{i}_1 \underline{j}_1, \ldots, \underline{g}_s \underline{h}_s \underline{i}_s \underline{j}_s | \underline{k}_1 \underline{l}_1 \underline{m}_1 \underline{n}_1 \underline{o}_1, \ldots, \underline{k}_t \underline{l}_t \underline{m}_t \underline{n}_t \underline{o}_t\}^{\pm}$$

$\underline{a}_1$		$\underline{a}_p$		$\underline{b}_1$			$\underline{b}_q$	$\underline{d}_1$		$\underline{d}_r$	$\underline{g}_1$		$\underline{g}_s$		$\underline{k}_1$		$\underline{k}_t$
$\underline{c}_1$							$\underline{c}_q$	$\underline{e}_1$		$\underline{e}_r$	$\underline{h}_1$		$\underline{h}_s$		$\underline{l}_1$		$\underline{l}_t$
								$\underline{f}_1$		$\underline{f}_r$	$\underline{i}_1$		$\underline{i}_s$		$\underline{m}_1$		$\underline{m}_t$
											$\underline{j}_1$		$\underline{j}_s$		$\underline{n}_1$		$\underline{n}_t$
															$\underline{o}_1$		$\underline{o}_t$

[p,0,0,0,0] + [0,q,0,0,0] + [0,0,r,0,0] + [0,0,0,s,s] +

[0,0,0,0,2t] with self-duality

[0,0,0,2t,0] with antiself-duality
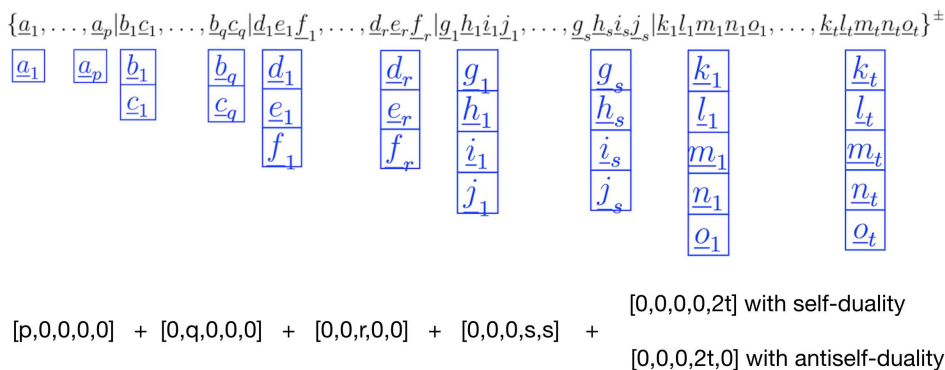
Figure 2: Young Tableaux-Index Structure Notation & Conventions # 1

where in Figure 2 we have "disassembled" the YT to show how each column is affiliated with each type of subscript structure. In Figure 3, we have assembled all the column into a proper YT.

$\underline{k}_1$	...	$\underline{k}_t$	$\underline{g}_1$	...	$\underline{g}_s$	$\underline{d}_1$	...	$\underline{d}_r$	$\underline{b}_1$	...	$\underline{b}_q$	$\underline{a}_1$	...	$\underline{a}_p$
$\underline{l}_1$	...	$\underline{l}_t$	$\underline{h}_1$	...	$\underline{h}_s$	$\underline{e}_1$	...	$\underline{e}_r$	$\underline{c}_1$	...	$\underline{c}_q$			
$\underline{m}_1$	...	$\underline{m}_t$	$\underline{i}_1$	...	$\underline{i}_s$	$\underline{f}_1$	...	$\underline{f}_r$						
$\underline{n}_1$	...	$\underline{n}_t$	$\underline{j}_1$	...	$\underline{j}_s$									
$\underline{o}_1$	...	$\underline{o}_t$												

IR,$\pm$

[p,q,r,s,s+2t] with self-duality

[p,q,r,s+2t,s] with anti-self-duality
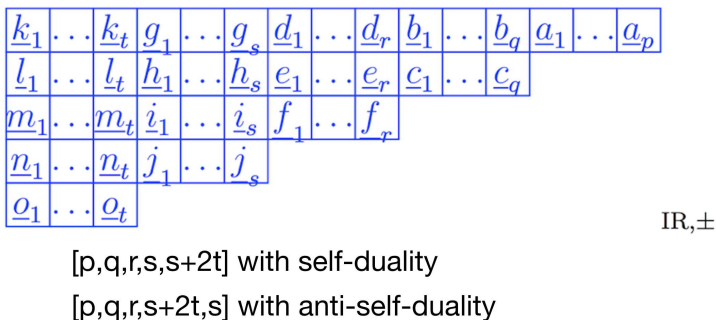
Figure 3: Young Tableaux-Index Structure Notation & Conventions # 2

As one moves from the YT's shown in Figure 2 to Figure 3, it is clear that the number of vertical boxes is tabulating the number of 1-forms, 2-forms, 3-forms, 4-forms, and 5-forms in the YT's. These are the entries between the vertical | bars. These precisely correspond to the Dynkin Labels p, q, r, s, and t. A first example of the correspondence between the subscript conventions and the affiliated YT and Dynkin Label is shown in (3.16).

More examples are provided in Appendix A.

$$(3.16) \qquad \{\underline{a}_2, \underline{a}_3 | \underline{a}_1 \underline{b}_1 \underline{c}_1 \underline{d}_1\} \quad \equiv \quad \begin{array}{|c|c|c|} \hline \underline{a}_1 & \underline{a}_2 & \underline{a}_3 \\ \hline \underline{b}_1 \\ \hline \underline{c}_1 \\ \hline \underline{d}_1 \\ \hline \end{array}_{\text{IR}} \quad \equiv \quad [2,0,0,1,1] \quad .$$

One remark is that the { }-indices that include five vertical boxes within them are separated into anti-self-dual and self-dual components that satisfy the equations

$$(3.17) \qquad \{\underline{a}_1 \underline{b}_1 \underline{c}_1 \underline{d}_1 \underline{e}_1\}^{\pm} = \pm \tfrac{1}{5!} \epsilon_{\underline{a}_1 \underline{b}_1 \underline{c}_1 \underline{d}_1 \underline{e}_1}{}^{\underline{a}_2 \underline{b}_2 \underline{c}_2 \underline{d}_2 \underline{e}_2} \{\underline{a}_2 \underline{b}_2 \underline{c}_2 \underline{d}_2 \underline{e}_2\}^{\pm} \quad .$$

Figure 3 is the $\mathfrak{so}(10)$ generalization of the one shown in Equation (3.1) for $\mathfrak{su}(3)$. However, there is one important difference. The YT itself shown in Equation (3.1) is related to irreducible representations of $\mathfrak{su}(3)$, while in Figure 3 the YT with "IR" subscript subject to certain conditions refers to irreducible representations of $\mathfrak{so}(10)$. YT's without this subscript are reducible with respect to $\mathfrak{so}(10)$. In the next chapter, we will deal with extracting irreducible representations of $\mathfrak{so}(10)$ from YT's without "IR" subscript by a graphical means.

3.4. Irreducibility conditions

Irreducible bosonic Young Tableaux only tell us the index structures of the fields when we translate the irrep descriptions into field variable language. If we want the correct d.o.f.[5] of the fields, we have to include the irreducibility conditions or constraints. The irreducibility conditions are effectuated by the branching rules for $\mathfrak{su}(10) \supset \mathfrak{so}(10)$. Examples are

$$(3.18) \qquad \Box = \Box_{\text{IR}} \ , $$
$$\{10\} = \{10\}$$

$$(3.19) \qquad \Box\Box = \Box\Box_{\text{IR}} \ \oplus \ \cdot $$
$$\{55\} = \{54\} \ \oplus \ \{1\} \ , $$

$$(3.20) \qquad \begin{array}{l}\Box\Box \\ \Box\end{array} = \begin{array}{l}\Box\Box \\ \Box\end{array}_{\text{IR}} \ \oplus \ \Box_{\text{IR}} \ , $$
$$\{330\} = \{320\} \ \oplus \ \{10\}$$

---

[5] We use d.o.f. as the abbreviation for "degrees of freedom."

$$(3.21) \quad \ytableausetup{} \quad \square = \square_{\text{IR},-} \oplus \square_{\text{IR},+} \oplus \square_{\text{IR}} \quad .$$

$$\{2310\} = \{1050\} \oplus \{\overline{1050}\} \oplus \{210\}$$

In each above equation, the leading term tells us the index structure corresponding to the irrep and the remaining terms tell us the irreducible conditions or constraints. The degrees of freedom contributed by each term are also presented below the YT. They can be translated into the field language respectively,

$$(3.22)$$

$$\Phi_{\{\underline{a}_1\}} : \quad \text{N/A} \quad ,$$

$$(3.23)$$

$$\Phi_{\{\underline{a}_1,\underline{a}_2\}} : \quad \eta^{a_1 a_2} \, \Phi_{\{\underline{a}_1,\underline{a}_2\}} = 0 \quad ,$$

$$(3.24)$$

$$\Phi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\}} : \quad \eta^{a_1 a_2} \, \Phi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\}} = 0 \quad ,$$

$$(3.25)$$

$$\Phi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}} : \begin{cases} \eta^{a_1 a_2} \, \Phi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}} = 0 \ , \\ \Phi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^+} = \frac{1}{5!}\epsilon_{\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1}{}^{\underline{f}_1\underline{g}_1\underline{h}_1\underline{i}_1\underline{j}_1} \, \Phi_{\{\underline{a}_2|\underline{f}_1\underline{g}_1\underline{h}_1\underline{i}_1\underline{j}_1\}^+} \ , \\ \Phi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^-} = -\frac{1}{5!}\epsilon_{\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1}{}^{\underline{f}_1\underline{g}_1\underline{h}_1\underline{i}_1\underline{j}_1} \, \Phi_{\{\underline{a}_2|\underline{f}_1\underline{g}_1\underline{h}_1\underline{i}_1\underline{j}_1\}^-} \ . \end{cases}$$

where

$$(3.26) \quad \Phi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}} = \Phi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^+} + \Phi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^-} \quad .$$

as indicated in the text around Equation (3.12).

In the next chapter, we will see how these branching rules can be obtained by graphical rules.

4. 10D bosonic young tableau tying rules

From Chapter 3, we know irreducible bosonic Young Tableaux for $\mathfrak{so}(10)$ can be drawn. In the $A_{n-1} = \mathfrak{su}(n)$ algebra, one can calculate the dimension of an irrep by the well-known graphical device involving the use of the "hook rule." This leads to the question whether there exists a *diagrammatic method*

to calculate the dimension of an irrep from the $\mathfrak{so}(10)$ BYT directly, instead of translating it to a Dynkin Label $[a_1, a_2, a_3, a_4, a_5]$ constructed from the five integers a_1, a_2, a_3, a_4, and a_5 followed by substituting the integers into the Weyl dimension formula for the D_5 algebra [7]

(4.1)
$$d(a_1, a_2, a_3, a_4, a_5)$$
$$= \Big[\prod_{k=1}^{3} \prod_{i=1}^{k} \Big(\frac{\sum_{j=i}^{k} a_j}{k - i + 1} + 1 \Big) \Big] \Big[\prod_{k=0}^{3} \prod_{l=4}^{5} \Big(\frac{(\sum_{j=1}^{k} a_{4-j}) + a_l}{k + 1} + 1 \Big) \Big]$$
$$\times \Big[\prod_{k=0}^{2} \prod_{i=k+1}^{3} \Big(\frac{(\sum_{j=k+1}^{i} a_{4-j}) + (\sum_{j=1}^{k} 2a_{4-j}) + a_4 + a_5}{i + k + 2} + 1 \Big) \Big]$$
$$= \Big[\frac{1}{7! \times 5! \times 4! \times 3!} \Big] (a_1 + 1)(a_2 + 1)(a_3 + 1)(a_4 + 1)(a_5 + 1)$$
$$(a_1 + a_2 + 2)(a_2 + a_3 + 2)(a_3 + a_4 + 2)(a_3 + a_5 + 2)$$
$$(a_1 + a_2 + a_3 + 3)(a_2 + a_3 + a_4 + 3)(a_2 + a_3 + a_5 + 3)(a_3 + a_4 + a_5 + 3)$$
$$(a_1 + a_2 + a_3 + a_4 + 4)(a_1 + a_2 + a_3 + a_5 + 4)(a_2 + a_3 + a_4 + a_5 + 4)$$
$$(a_2 + 2a_3 + a_4 + a_5 + 5)(a_1 + a_2 + a_3 + a_4 + a_5 + 5)$$
$$(a_1 + a_2 + 2a_3 + a_4 + a_5 + 6)(a_1 + 2a_2 + 2a_3 + a_4 + a_5 + 7) \quad .$$

On the other hand, we already know there exist branching rules $\mathfrak{su}(10) \supset \mathfrak{so}(10)$ such that a $\mathfrak{su}(10)$ irrep can be projected to a direct sum of irreps in $\mathfrak{so}(10)$. In a branching rule, one of the irreps in $\mathfrak{so}(10)$ must have the same YT shape with that in $\mathfrak{su}(10)$. Also, the dimensions of the totality of representations obtained from the branching, when added together, should match that of the "unbranched" representation. These properties manifest in the examples of the analytical expressions of irreducible conditions as mentioned in Section 3.4.

This inspires us to invent a diagrammatic method to obtain the branching rules of $\mathfrak{su}(10) \supset \mathfrak{so}(10)$. Meanwhile, requiring the dimensions to match before and after projections would give us dimensions of BYT in $\mathfrak{so}(10)$.

This diagrammatic method is called tying rule. The origin of the rule's name comes from the notion of "ties". Given two symmetric boxes in a BYT, one can "tie" them by putting a node $\bullet$ in each box and a line between them. In a field theoretical formulation using index notation, this means contracting the two symmetric indices with the flat metric $\eta^{\underline{ab}}$, i.e. taking a "trace". The degrees of freedom are thus equivalent to those in the diagram where those two boxes are "eliminated". This is illustrated in Equation (4.2),

where on the left the YT has two symmetric boxes, and on the right the "tying" action is applied on them.

(4.2)
$$\begin{array}{c} \boxed{\underline{a}\,|\,\underline{b}} \\[4pt] \Phi_{\{\underline{a},\underline{b}\}} \end{array} \qquad\qquad \begin{array}{c} \boxed{\bullet\!-\!\!\bullet} \equiv \cdot \\[4pt] \eta^{\underline{ab}}\,\Phi_{\{\underline{a},\underline{b}\}} \equiv \varphi \end{array}$$

In the following, we will express everything about the tying rule graphically without mentioning the η-contractions, but readers should keep this in mind.

For completely antisymmetric BYTs, since no tie can be drawn (no two symmetric boxes), the rules for calculating the dimensions are exactly the same as the YTs in $\mathfrak{su}(10)$. We will introduce the tying rules in the order of completely symmetric and two-column (Section 4.2), three-column (Section 4.3), and n-column ($n \geq 4$) BYTs (Section 4.4), where the latter ones in the list have more complications and the former ones are specific cases of them. While the tying rules up till 3-column BYTs are complete, tying rules for 4-column and above remain incomplete as only several specific cases are verified. This requires further investigation beyond this paper.

A remark is tying rule also works for general $\mathfrak{su}(N) \supset \mathfrak{so}(N)$, i.e. $A_{N-1} \supset D_{N/2}$ for even N or $A_{N-1} \supset B_{(N-1)/2}$ for odd N[6]. The only difference between branching into a D-series algebra and a B-series algebra is that the BYTs containing column(s) of $N/2$ boxes in the $D_{N/2}$ algebra can be split into a self-dual irrep and an anti-self-dual irrep (and thus contain two irreps), but none of the BYTs in the B-series algebra have these self-duality properties (so they will never be split into two irreps). A little Mathematica program available in GitHub is written to verify the tying rules for all the BYTs up to 3-columns for $N > 6$.

Another remark is that based on tying rules we can also calculate the dimensionalities of bosonic irreps which will be presented in the following sections. One can quickly check the consistency of the Weyl dimension formula (4.1), the $\mathfrak{su}(10)$ algebra hook rule, and our graphical tying rules. A

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[6] It seems likely that the proposal could be extended, with appropriate modifications to cases relating branchings of the $\mathfrak{su}(N)$ into the $\mathfrak{usp}(N)$ algebras (i.e. A_{2N-1} into C_N) as the latter possess a quadratic symplectic invariant. In this case, the tying would take place vertically not horizontally. In [35], King introduced two set of step-by-step graphical rules to describe $\mathfrak{su}(N) \supset \mathfrak{so}(N)$ and $\mathfrak{su}(N) \supset \mathfrak{usp}(N)$ branching rules respectively. The former is an interpretation of Littlewood's rule (that would be described later in Section 4.1), and one would see later how similar in spirit it is to the tying rule. We propose that the latter set of graphical rules by King is in fact in the same vein with the vertical tying rule, with the exact same mapping between Littlewood's rule and the horizontal tying rule.

simple python code to calculate Weyl dimension formula by inputting the Dynkin Label can be found in GitHub.

But before we dive into the tying rule, we want to review a rule first proposed by Littlewood [33, 35] and we propose as a conjecture[7] that our tying rule (up to 3-columns) must be equivalent. The rule in question is carried out by utilizing a specific Schur function series and a "division operator" of Young Tableaux. We will present the rule in Section 4.1.

4.1. Littlewood's rules by a Schur function series

Schur functions were introduced by Issai Schur (1875 - 1941) in his doctoral dissertation in 1901 [27]. They are symmetric functions that are more general than elementary symmetric functions, which their symmetry properties can be represented by Young Tableaux. In 1934, Littlewood and Richardson published the famous Littlewood-Richardson rule [28] that gives the combinatorial description of the coefficients in the decompositions of the product of two Schur functions as a linear combination of other Schur functions. It was then broadly applied in the area of classical Lie algebras to understand tensor products between two irreducible representations. For physicists, there are also familiar books [38, 39] where such introductions and discussions appear.

If we denote an irreducible representation of one of the A_n algebras by $\{\lambda\}$, where $\{\ \}$ means A_n and λ indicates the Young Tableau shape, we can write the Littlewood-Richardson rule as

$$(4.3) \qquad \{\lambda\} \otimes \{\mu\} \; = \; \sum_\nu m^\nu_{\lambda\mu} \{\nu\} \quad .$$

where $m^\nu_{\lambda\mu}$ is a number. The way to obtain the tensor product is familiar: put "a" labels in the first row of $\{\mu\}$, "b" labels in the second, "c" labels in the third, etc., and exploit all combinations of sticking these boxes containing these letters to the YT $\{\lambda\}$ such that they obey the rules,

1) The final Young diagram is regular (has a standard shape, not skew a shape);

2) No two of the same letters sit in the same column (so that symmetry is not violated); and

---

[7] A rigorous mathematical proof is currently lacking, but in the following sections we present supporting evidence.

3) When going from left to right, top to bottom, the accumulated number of "a" is always greater than that of "b", and that of "b" is always greater than that of "c", etc..

The coefficients $m^\nu_{\lambda\mu}$ can then be obtained by counting the numbers of $\{\nu\}$ in the final decomposition.

We can then turn to a related operation, called quotient operation "/"[8], which is defined by

$$(4.4) \qquad \{\nu\}/\{\mu\} \;=\; \sum_\lambda m^\nu_{\lambda\mu}\{\lambda\} \quad,$$

where $m^\nu_{\lambda\mu}$ are exactly the same as the coefficients of the Littlewood-Richardson rule for products [29–31]; and $\{\nu\}/\{\mu\}$ is called a skew Schur function.

The coefficients above immediately suggest that the same rules 1 - 3 stated above apply to the quotient operation, but now we put letters from $\{\mu\}$ into $\{\nu\}$ and eliminate those boxes with letters. An example of the quotient operation would be

_____

[8] Note that it is not the inverse operation of the Kronecker product $\otimes$.

The meaning of these coefficients for quotient are the same as those for product, illustrated by this example, is

(4.6)

More explicitly, if we let

(4.7)

according to the notations in Equations (4.3) and (4.4), then we have the coefficients

$$(4.8) \quad m^{\nu}_{\lambda_1 \mu} = 1 \quad , \quad m^{\nu}_{\lambda_2 \mu} = 1 \quad , \quad m^{\nu}_{\lambda_3 \mu} = 2 \quad , \quad m^{\nu}_{\lambda_4 \mu} = 1 \quad .$$

With these understandings of the quotient operation, we can proceed to the statement of the algorithm of calculating the branching rules of $\mathfrak{su}(n) \supset \mathfrak{so}(n)$. By manipulating Schur function series, Littlewood proved [33]

$$(4.9) \qquad \mathfrak{su}(n) \supset \mathfrak{so}(n) \; : \; \{\lambda\} \supset [\lambda/D] \quad ,$$

where [ ] means a representation in $\mathfrak{so}(n)$ algebra, and

$$(4.10) \quad D \;=\; \cdot \;\oplus\; \boxed{}_{\text{IR}} \;\oplus\; \boxed{}_{\text{IR}} \;\oplus\; \ydiagram{2,2}_{\text{IR}}$$

$$\oplus\; \boxed{}_{\text{IR}} \;\oplus\; \ydiagram{3,2}_{\text{IR}} \;\oplus\; \ydiagram{2,2,2}_{\text{IR}} \;\oplus\; \cdots$$

is a Schur function series[9] written in our notation, which contains all partitions with all parts even. King in his 1971 paper [35] understood this rule diagrammatically through the rules of obtaining quotients, like those in Equation (4.5), by quotient-ing all the terms in D. He described this intuition as "step-by-step nature of the trace removal process of $\mathfrak{so}(n)$", which echoes with our definition of "tie" in Equation (4.2) but we are very explicit in the symmetric property of the η-metric by putting the two nodes on the same row.

For clarity, let us apply Equation (4.9) to

$$(4.11) \qquad\qquad \lambda \;=\; \ydiagram{3,2,1} \qquad .$$

By implementing quotient operator with each term in D, only the terms below give non-vanishing results,

---

[9] This Schur function series D contains all possible YTs constructed from $\boxed{}_{\text{IR}}$. Therefore, the number of terms containing $2n$ box(es) is the integer partition of n. This is why the Littlewood's rule is very similar to tying rule - that "$/D$" means removing all the combinations of $\boxed{}_{\text{IR}}$, while tying rules involve removing all the combinations of tied boxes $\boxed{\bullet\!-\!\bullet}_{\text{IR}}$.

(4.12)

while other terms vanish. This leads to the result

$$\{235950\} \ = \ \{174636\} \ \oplus \ \{17920\} \ \oplus \ \{7644\}$$

$[1,2,1,0,0] \qquad [1,1,1,0,0] \qquad [0,3,0,0,0]$

(4.13)

$[3,0,1,0,0] \qquad\qquad [2,2,0,0,0]$

$$\oplus \ \{14784\} \ \oplus \ \{16380\}$$

$$\oplus \ 2 \ \square\square\square \ \oplus \ \square\square\square \ \oplus \ \square\square\square \ \oplus \ \square\square \ \oplus \ \square\square$$

$$[2,1,0,0,0] \qquad [1,0,1,0,0] \quad [0,2,0,0,0] \ [0,1,0,0,0] \ [2,0,0,0,0]$$

$$\oplus \quad (2)\{1386\} \quad \oplus \quad \{945\} \quad \oplus \quad \{770\} \quad \oplus \quad \{45\} \quad \oplus \quad \{54\}$$

.

4.2. Tying rule for completely symmetric and two-column BYTs

Now we turn to tying rule. For completely symmetric and two-column bosonic Young Tableaux, there are two main steps to find the branching rules. Given a YT in $\mathfrak{su}(10)$, we perform the following to find how it decomposes into a direct sum of irreps in $\mathfrak{so}(10)$.

Step 1:: Draw all possible combination of ties (including no tie at all). Vertical position of ties does not matter. Keep one copy for each of the equivalent ones.

Step 2:: For each diagram, wipe out the boxes with ties. Then the decomposition in $\mathfrak{so}(10)$ would be the sum of all these BYTs.

A very detailed description of this algorithm is given below to foster a thorough understanding about the meaning of the above two main steps. Let n be the number of boxes of the Young Tableau to be decomposed.

1) Draw a tie between two boxes in the same row. That is equivalent to contracting two vector indices with a metric. Next we erase those two boxes and create a new Young Tableau (with $n-2$ boxes) in the decomposition.

2) Repeat the above step until no tie can be drawn further. In each step we create a new Young Tableau with $n-2t$ boxes, where t is the number of ties. Then the entire Young Tableau can be decomposed into direct sums of $t_{\max}+1$ irreducible Young Tableaux (with subscript IR as we denote them).

3) Look at the Young Tableau with the maximum number of ties (or the minimum number of boxes without ties). Calculate the dimension using the usual Young Tableau rules and call it $d_{t_{\max}}$.

4) Look at the Young Tableau with the next maximum number of ties (or the next minimum number of boxes without ties). Calculate the dimension using the usual Young Tableau rules and call it $\tilde{d}_{t_{\max}-1}$.

Then the dimension of this irreducible Young Tableau is $d_{t_{\max}-1} = \tilde{d}_{t_{\max}-1} - d_{t_{\max}}$.

5) Repeat the above step for all irreducible Young Tableaux until $t = 0$. For a general t, the dimension of that irreducible Young Tableau is $d_t = \tilde{d}_t - \sum_{t'>t} d_{t'}$, where $\tilde{d}_t$ is the dimension calculated by the usual Young Tableau rules, and $d_{t'}$ are the dimensions of the irreducible representations.

6) Finally, we obtain a decomposition of the reducible Young Tableau into a direct sum of irreducible Young Tableaux. The dimensions satisfy $\tilde{d}_0 = d_0 + d_1 + \cdots + d_{t_{\max}-1} + d_{t_{\max}}$.

To make things as clear as possible, we discuss two explicit examples. The first example is a completely symmetric Young Tableau. Steps 1 - 2 allow us to draw the ties and the irreducible Young Tableaux as follows.

(4.14)

This is the branching rule for in $\mathfrak{su}(10)$ to $\mathfrak{so}(10)$. So $t_{\max} = 2$, and thus the final decomposition has $t_{\max} + 1 = 3$ irreducible Young Tableaux. In this example, $n = 5$, and the three irreducible Young Tableaux have 5, 3, and 1 box(es) respectively. Steps 3 - 5 allow us to find the dimensions of the irreducible Young Tableaux as follows.

(4.15)

$$d_2 = 10 \quad,$$

$$d_1 = \tilde{d}_1 - d_2 = \frac{10 \times 11 \times 12}{3 \times 2 \times 1} - 10 = 210 \quad,$$

$$d_0 = \tilde{d}_0 - d_1 - d_2$$
$$= \frac{10 \times 11 \times 12 \times 13 \times 14}{5 \times 4 \times 3 \times 2 \times 1} - 210 - 10 = 1782 \quad.$$

Therefore,

(4.16)
$$\{2002\} = \{1782\} \oplus \{210'\} \oplus \{10\} \quad,$$

where the irreducible representation $\{2002\}$ of $\mathfrak{su}(10)$ corresponds to the Dynkin Label $[5,0,0,0,0,0,0,0,0,0]$, while $\{1782\}$, $\{210'\}$ and $\{10\}$ of $\mathfrak{so}(10)$ corresponds to the Dynkin Labels $[5,0,0,0,0]$, $[3,0,0,0,0]$ and $[1,0,0,0,0]$ respectively.

The second example is a two-column Young Tableau. Steps 1 - 2 allow us to draw the ties and the irreducible Young Tableaux as follows.

(4.17)

This is the branching rule for in $\mathfrak{su}(10)$ to $\mathfrak{so}(10)$. Again $t_{\max} = 2$, so we have 3 irreducible Young Tableaux in the decomposition. And again $n = 5$, so we have three irreducible Young Tableaux with 5, 3, and 1 box(es) respectively. Steps 3 - 5 allow us to find the dimensions of the irreducible Young Tableaux as follows.

(4.18)

$$d_2 = 10 \quad ,$$

$$d_1 = \tilde{d}_1 - d_2 = \frac{10 \times 11 \times 9}{3 \times 1 \times 1} - 10 = 320 \quad ,$$

$$d_0 = \tilde{d}_0 - d_1 - d_2 = \frac{10 \times 11 \times 9 \times 10 \times 8}{4 \times 2 \times 3 \times 1 \times 1} - 320 - 10 = 2970 \quad .$$

Therefore,

(4.19) $\{3300\} = \{2970\} \oplus \{320\} \oplus \{10\} \quad ,$

where the irreducible representation $\{3300\}$ of $\mathfrak{su}(10)$ corresponds to the Dynkin Label $[0,1,1,0,0,0,0,0,0]$, while $\{2970\}$, $\{320\}$ and $\{10\}$ of $\mathfrak{so}(10)$ corresponds to the Dynkin Labels $[0,1,1,0,0]$, $[1,1,0,0,0]$ and $[1,0,0,0,0]$ respectively.

4.3. Tying rule for three-column BYTs

For a three-column BYT, there are two steps to figure out how a $\mathfrak{su}(10)$ irrep (which is reducible in $\mathfrak{so}(10)$) decomposes into $\mathfrak{so}(10)$ irreducible parts.

Step 1:: Draw all the possible combinations of ties (including no tie at all). They can skip a box (as long as they are on the same row and remain symmetric). Vertical position of ties does not matter. Keep one copy for the equivalent ones.

Step 2:: For each diagram, wipe out the boxes with the ties and rearrange them vertically *according to symmetry properties* (we'll explain it shortly) such that they have standard YT shapes. If a standard YT shape cannot be obtained, or the rearrangements cannot be performed with symmetry preserved, throw that BYT away. Then the decomposition in $\mathfrak{so}(10)$ would be the sum of all the remaining BYTs.

Let us focus on step 1 first. To illustrate the idea, we will discuss two examples below.

(4.20)

(4.21)
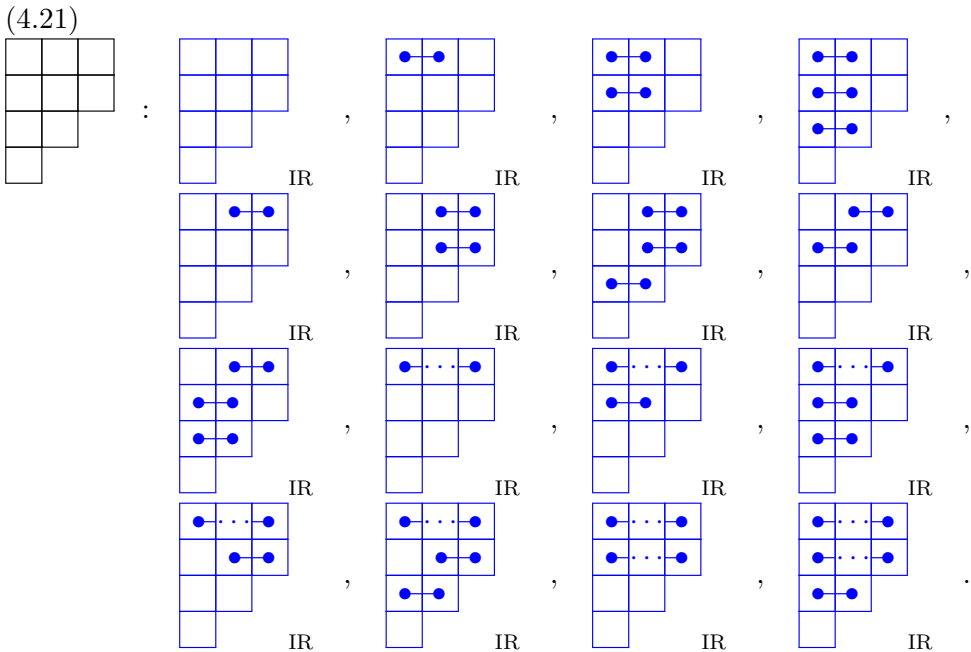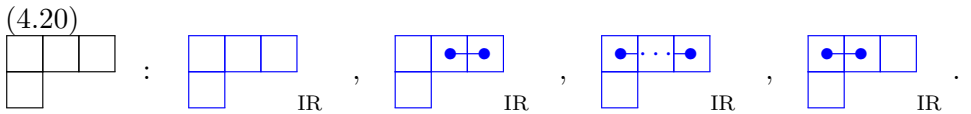
Now, let us consider step 2. Before we proceed with examples, let us explain in detail the meaning of the symmetry properties we mentioned above. In general, consider a YT containing two columns of unequal numbers of boxes, one with M boxes and one with N boxes. Without loss of generality,

let $M < N$.

(4.22)

$$N \left\{ \vphantom{\begin{matrix} \\ \\ \\ \\ \end{matrix}} \right. \begin{matrix} \boxed{\vdots \ \vdots} \end{matrix} \left. \vphantom{\begin{matrix} \\ \\ \end{matrix}} \right\} M$$

The symmetry relations go as follows.

1) The i-th box in the M-column is *symmetric* with the i-th box in the N-column $(i = 1, \ldots, M)$;

2) The i-th box in the M-column is *antisymmetric* with the j-th box in the N-column for $j \neq i$ and $j \leq M$ $(j = 1, \ldots, M)$;

3) The i-th box in the M-column has *no symmetry relation* with the k-th box in the N-column for $k > M$ $(k = M + 1, \ldots, N)$.

If we use the example in Equation (4.21), we can illustrate these three conditions as follows.

(4.23)

symmetric	antisymmetric	no symmetry relation

We have explained the definitions of the symmetry relations between any pair of boxes in a BYT. What does it mean by saying the symmetry properties are *preserved*? Since we perform all the ties horizontally, we are only allowed to move the boxes vertically. Then three scenarios below are possible after rearrangements.

(4.24)

must	must not	no restriction

Symmetry properties are *preserved* if the final BYT abides by these rules, i.e. the originally symmetric boxes must remain symmetric, the originally antisymmetric boxes must not become symmetric, and the boxes that do not have any symmetry relation originally can end up being symmetric.

With these in mind, we can now erase the boxes that are tied up (with a bullet in the middle), and rearrange them vertically. Let us start with

the first example in Equation (4.20). The last diagram can not be vertically rearranged to be a regular YT shape,

(4.25)

$$\text{(diagram)}_{\text{IR}} \longrightarrow \square \quad \square \quad .$$

The remaining diagrams could be rearranged as follows,

(4.26)

$$\text{(diagram)}_{\text{IR}} \longrightarrow \text{(diagram)}_{\text{IR}} \quad ,$$

$$\text{(diagram)}_{\text{IR}} \longrightarrow \boxed{\sim}\boxed{\sim}_{\text{IR}} \quad .$$

Both of these diagrams are allowed as they have standard YT shapes and all the symmetry properties are preserved. Together with the diagram with no tie, we can write the branching rule

(4.27)

$$\text{(diagram)} = \text{(diagram)}_{\text{IR}} \oplus \text{(diagram)}_{\text{IR}} \oplus \text{(diagram)}_{\text{IR}} \quad .$$
$$[2,1,0,0,0] \quad [0,1,0,0,0] \quad [2,0,0,0,0]$$
$$\{1485\} = \{1386\} \oplus \{45\} \oplus \{54\}$$

We turn to the more complicated second example in Equation (4.21). Before we perform delicate vertical rearrangements, we can discard diagrams that obviously could not be arranged to standard YT shapes, including

(4.28)

$$\text{(diagram)}_{\text{IR}} \longrightarrow \text{(diagram)} \quad , \quad \text{(diagram)}_{\text{IR}} \longrightarrow \square \ \square \quad ,$$

$$\text{(diagram)}_{\text{IR}} \longrightarrow \square \ \square \quad , \quad \text{(diagram)}_{\text{IR}} \longrightarrow \text{(diagram)} \quad ,$$

$$\text{(diagram)}_{\text{IR}} \longrightarrow \text{(diagram)} \quad .$$

We can now carry out the rearrangements of the remaining diagrams. To facilitate the presentation, in the rearrangement process, we will use the

number "1" to indicate the boxes in the first row of the final diagram, and the number "2" to indicate those in the second row, etc.. We will use the colors in Equations (4.23) and (4.24) to indicate the conditions of symmetric, antisymmetric, and no symmetry relation respectively. For a row with a single box, we use black color. Since it's always about pairs of boxes, some boxes might need to be indicated twice. To avoid this, we will specify the conditions of symmetric and antisymmetric first, then if the box next to it is in green color, it has no symmetry relation with the other boxes on the same row. Here are the ones that are allowed according to symmetry rules.

(4.29)

The remaining diagram is

(4.30)

which is not allowed by the symmetry rule in Equation (4.24), and thus disregarded. Therefore,

(4.31)

$$\{304920\} = \{192192\} \oplus \{34398\} \oplus \{36750\}$$

IR $[0,1,1,1,1]$ IR $[0,2,1,0,0]$ IR $[1,1,0,1,1]$

$$\oplus \{1728\} \oplus \{120\} \oplus \{4312\} \oplus \{27720\}$$

IR $[1,0,0,1,1]$ IR $[0,0,1,0,0]$ IR $[2,0,1,0,0]$ IR $[1,0,2,0,0]$

$$\oplus \{2970\} \oplus \{4410\} \oplus \{320\}$$

IR $[0,1,1,0,0]$ IR $[1,2,0,0,0]$ IR $[1,1,0,0,0]$

To foster a better feeling for these rules, especially about the diagrams that violate the symmetry properties, i.e. contain the antisymmetric boxes on the same row in Equation (4.24), we put a very enlightening example at the end. Since it's very easy to observe which BYT would not end up being a standard shape, those would be excluded right away. The equivalent ones would not be written twice also.

(4.32)

Now, let us just focus on those diagrams that would inevitably contain two antisymmetric boxes in one row, and thus must be discarded.

(4.33)

(4.34)

From the way these two diagrams vanish, we know there are 4 more diagrams that vanish in the same way,

(4.35)

(4.36)

We call the two diagrams in Equation (4.35) the *descendants* of the diagram in Equation (4.33), as both diagrams contain the pair of ties (on the top

two rows) of the (4.33) diagram which caused the diagram to contain antisymmetric boxes on the same row; and therefore the descendants would contain these exact same antisymmetric boxes and be discarded for the exact same reason. In a similar sense, the two diagrams in Equation (4.36) are the descendants of the diagram in Equation (4.34) as they contain the same problematic triplet of ties (on the top three rows) of the (4.34) diagram. Thus the final decomposition is

$$
\begin{aligned}
\{1176120\} \;=\; &
\begin{array}{c}
\text{IR} \\
[0,0,0,4,2] \\
\oplus\,[0,0,0,2,4] \\
\{141570\} \\
\oplus\,\{\overline{141570}\}
\end{array}
\;\oplus\;
\begin{array}{c}
\text{IR} \\
[0,0,2,2,0] \\
\oplus\,[0,0,2,0,2] \\
\{144144'\} \\
\oplus\,\{\overline{144144'}\}
\end{array}
\;\oplus\;
\begin{array}{c}
\text{IR} \\
[0,2,0,2,0] \\
\oplus\,[0,2,0,0,2] \\
\{46800\} \\
\oplus\,\{\overline{46800}\}
\end{array} \\[2ex]
\oplus\; &
\begin{array}{c}
\text{IR} \\
[2,0,0,2,0] \\
\oplus\,[2,0,0,0,2] \\
\{4950\} \\
\oplus\,\{\overline{4950}\}
\end{array}
\;\oplus\;
\begin{array}{c}
\text{IR} \\
[0,0,0,2,0] \\
\oplus\,[0,0,0,0,2] \\
\{126\} \\
\oplus\,\{\overline{126}\}
\end{array} \\[2ex]
\oplus\; &
\begin{array}{c}
\text{IR} \\
[0,0,1,2,2] \\
\oplus\,\{270270\}
\end{array}
\;\oplus\;
\begin{array}{c}
\text{IR} \\
[0,1,1,1,1] \\
\oplus\,\{192192\}
\end{array}
\;\oplus\;
\begin{array}{c}
\text{IR} \\
[1,1,0,1,1] \\
\oplus\,\{36750\}
\end{array}
\;\oplus\;
\begin{array}{c}
\text{IR} \\
[1,0,0,1,1] \\
\oplus\,\{1728\}
\end{array}
\end{aligned}
\tag{4.37}
$$

4.4. Tying rule for n-column BYTs ($n \geq 4$)

For a n-column BYT with $n \geq 4$, the tying rule is as follows.

Step 1: Draw all the possible combinations of ties (including no tie at all). They can skip any number of box(es) (as long as they are on the same row and remain symmetric). Vertical position of ties does not matter. *Keep one copy for each of the equivalent ones.*

Step 2: For each diagram, erase the boxes with the ties and rearrange them vertically according to symmetry properties such that they become a standard YT shape. If a standard YT shape cannot be obtained, or the rearrangements cannot be performed with symmetry preserved, throw that BYT away. Then the decomposition in $\mathfrak{so}(10)$ would be the sum of all the remaining BYTs.

In the $\mathfrak{su}(10) \supset \mathfrak{so}(10)$ branching rules, when a Young Tableau in $\mathfrak{su}(10)$ contains 4 or more columns, the irreps in $\mathfrak{so}(10)$ begin to have multiplicities larger than one. The key to understanding 4+ columns is therefore in Step 1: what does *equivalence* mean when ties are drawn?

Notice that when a YT has 4 or more columns, there is *more than one way to put two ties in a row*. For example, two ties can be put in a row of 4 boxes in the following 3 different ways,

$$(4.38) \qquad \boxed{\bullet\!-\!\bullet\;\bullet\!-\!\bullet} \quad , \quad \boxed{\bullet\;\bullet\!\cdots\!\bullet\;\bullet} \quad , \quad \boxed{\bullet\;\bullet\!\cdots\!\bullet\;\bullet} \quad ,$$

given that these 4 boxes are inequivalent. The question now comes - when will they become equivalent? Stated another way, could we exchange two nodes that separately belong to two ties on the same row, such that the diagram after operation is equivalent to the original diagram? Let us remark that in these exchanges the specific boxes tied would not change, therefore they give the same irrep and it is for counting the multiplicity of each irrep in $\mathfrak{so}(10)$ after the application of branching rules.

To answer this question, let us study some special cases. Recall that in Section 4.2, the totally symmetric BYT example (4.14) does not contain 3 copies of $\boxed{}_{\text{IR}}$ but only one, as all of the five boxes are equivalent - they are all exchangeable (totally symmetric). So let us propose a condition of exchanging two nodes on two ties that would leave the diagram invariant.

1) When two nodes are in two columns with the same number of boxes - so the two boxes that the two nodes posited are equivalent and exchangeable.

For the special class of BYTs with all columns having the same number of box(es), this rule must apply. An example other than (4.14) is

(4.39)

Therefore,

$$[0, 4, 0, 0, 0] \qquad [2, 2, 0, 0, 0] \qquad [4, 0, 0, 0, 0]$$

$$\{70785\} \quad = \quad \{52920\} \quad \oplus \quad \{16380\} \quad \oplus \quad \{660\}$$

(4.40)

$$[0, 2, 0, 0, 0] \quad [2, 0, 0, 0, 0] \quad [0, 0, 0, 0, 0]$$

$$\oplus \quad \{770\} \quad \oplus \quad \{54\} \quad \oplus \quad \{1\}$$

In fact, for all BYTs with all columns having the same number of box(es), according to rule 1., all the irreps in the decompositions appear only once.

Now, let us turn to another class of BYTs - those with all columns unequal. Rule 1. would not apply. Although all columns are inequivalent, the

diagrams involving two ties on the same row do not naively have multiplic-
ities 3. We thereby propose the second condition of exchanging two nodes
that could make two diagrams equivalent.

2) When there is another tie connecting the two columns where the two
   nodes located - then this tie offers an additional symmetry.

Let us demonstrate this rule with the following example.

(4.41)
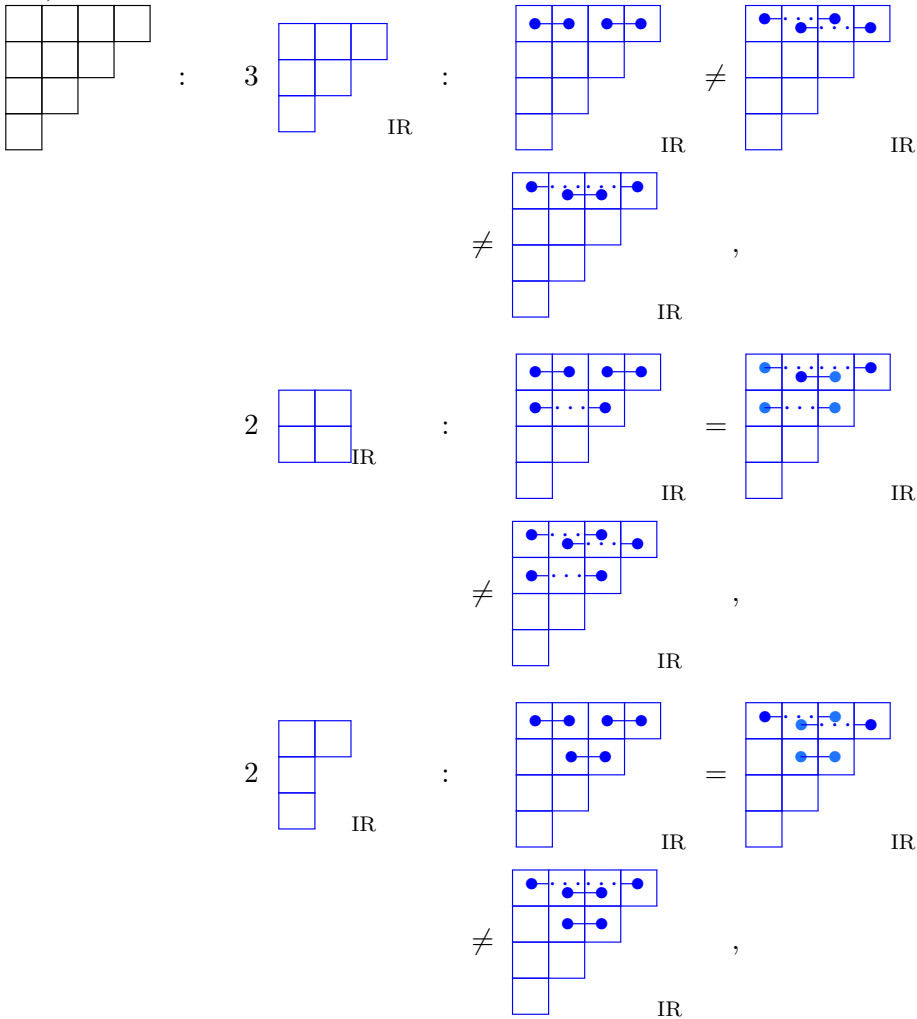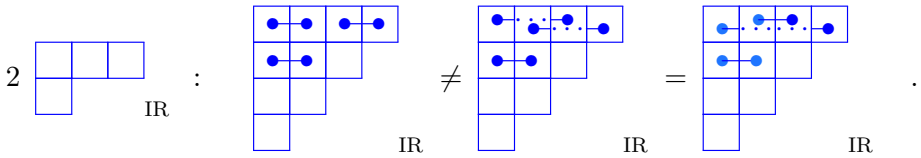
These are the irreps with multiplicities not equal to one. Others have multiplicity one. Some examples are

(4.42)
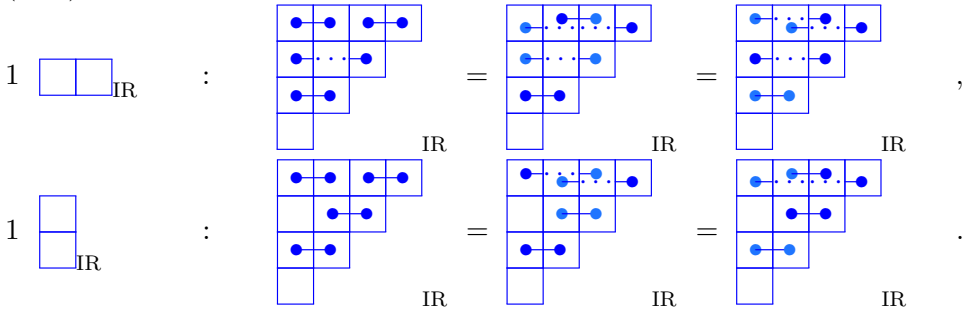
Other symmetry rules in Step 2 are exactly the same as those stated in 3-column BYTs (Section 4.3). Together we have

$\{1812096\}$ = $\{1048576\}$ ⊕ $\{174636\}$ ⊕ $\{143000\}$

⊕ $\{112320\}$ ⊕ $\{73710\}$ ⊕ $\{72765\}$

⊕ $\{70070\}$ ⊕ $(3)\{17920\}$ ⊕ $\{16380\}$
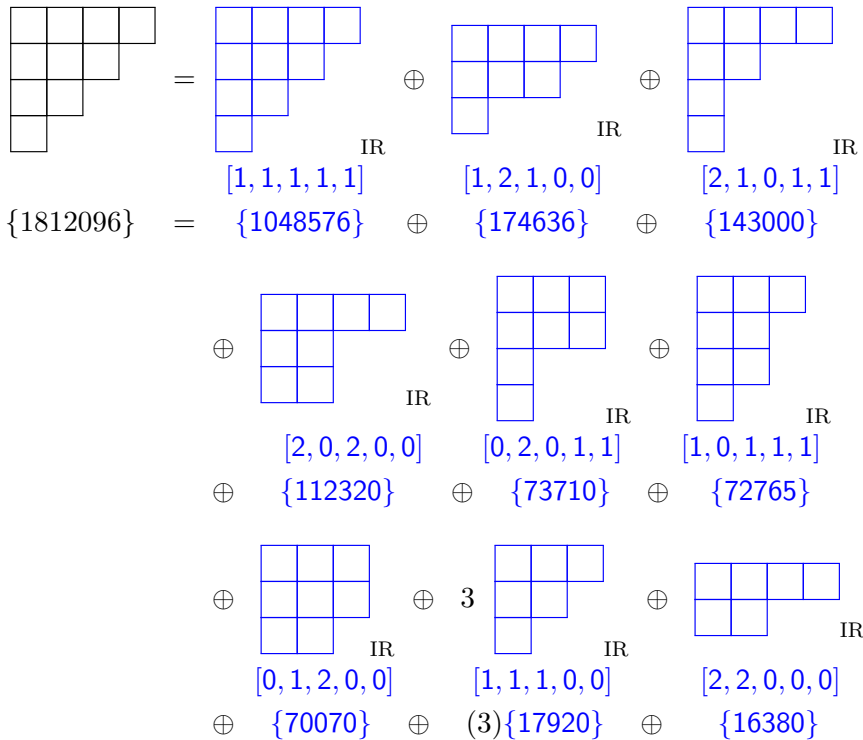
$$
\begin{aligned}
&\oplus \ \text{IR} \quad\quad \oplus \ \text{IR} \quad\quad \oplus \ \text{IR} \\
&\quad [3,0,1,0,0] \quad\ [2,0,0,1,1] \quad [0,3,0,0,0] \\
&\oplus \quad \{14784\} \quad \oplus \quad \{8085\} \quad \oplus \quad \{7644\}
\end{aligned}
$$

$$
\begin{aligned}
&\oplus \ \text{IR} \quad \oplus \ \text{IR} \quad \oplus \ 2 \ \text{IR} \quad \oplus \ 2 \ \text{IR} \\
&[0,1,0,1,1] \ \ [0,0,2,0,0] \quad\quad [2,1,0,0,0] \quad\quad [1,0,1,0,0] \\
&\oplus \ \{5940\} \quad \oplus \ \{4125\} \quad \oplus \quad (2)\{1386\} \quad \oplus \quad (2)\{945\}
\end{aligned}
$$

$$
\begin{aligned}
(4.43) \quad &\oplus \ 2 \ \text{IR} \quad \oplus \ \text{IR} \quad \oplus \ \text{IR} \\
&[0,2,0,0,0] \quad [2,0,0,0,0] \ \ [0,1,0,0,0] \\
&\oplus \ (2)\{770\} \quad \oplus \quad \{54\} \quad \oplus \quad \{45\}
\end{aligned}
$$

.

Now we report the status of those BYTs with some equal columns and some unequal columns. Although rules 1. and 2. apply to count most of the diagrams in the decompositions correctly, there are often one or two diagram(s) in each of those BYTs with multiplicities off by one. We hereby make a guess that there is a missing piece in these rules of exchanging nodes to count equivalences, and this is under investigation.

We close this section by restating the inspiration for our "tying rules" comes from the conventional physicist's approach to "pulling out" irreducible representations of tensors defined over orthogonal groups. Namely, given a general symmetrical tensor that is reducible, and an invariant quadratic tensor, the physicist will typically "pull out the irreps" by contracting indices on the symmetric tensor with the invariant quadratic tensor. The contraction of indices "ties" a pair of indices together, hence "tying rules."

5. Spinorial irreps of $\mathfrak{so}(10)$

In this chapter, we will exploit the graphical interpretation of Dynkin Labels of spinorial irreducible representations of $\mathfrak{so}(10)$ and its utility in translating

an irrep to field-theoretical language. This can be applied to understand the following two aspects of a spinorial irrep:

(a.) the proper index structure of the field; and

(b.) the irreducible constraints and the counting of the degrees of freedom.

These understandings of spinorial irreps are tightly related to some tensor product rules. They are presented in Chapter 6.

Let us start with a general explanation for (a.). The basic elements of spinorial irreps are

(5.1)
$$[0,0,0,0,1] \equiv \boxed{16} = \{16\} \ ,$$
$$[0,0,0,1,0] \equiv \boxed{\overline{16}} = \{\overline{16}\} \ .$$

For convenience, we introduce a notation for mixed YT that is defined from the Dynkin Label. Since putting together the bosonic columns in Equation (3.11) corresponds to adding Dynkin Labels, we can carry out a similar thing for these spinorial YT too. If we put together Equations (5.1) and (3.11), we can draw a general mixed YT[10]

(5.2)

$$= \begin{cases} [p,q,r,s,s+2t+1], & \text{for } + \text{ and } \boxed{} = \boxed{16} \ , \\ [p,q,r,s+2t,s+1], & \text{for } - \text{ and } \boxed{} = \boxed{16} \ , \\ [p,q,r,s+1,s+2t], & \text{for } + \text{ and } \boxed{} = \boxed{\overline{16}} \ , \\ [p,q,r,s+2t+1,s], & \text{for } - \text{ and } \boxed{} = \boxed{\overline{16}} \ . \end{cases}$$

One point to note though, is $\boxed{}$ and $\boxed{}$ are two different types of YT, so when they are put together on a row, it *does not* imply that they are symmetric. Also, the above Dynkin Labels should all be red as they are

---

[10] We represent the basic spinorial irrep by $\boxed{}$ and put it on the right of the BYT. In [13, 14], they represent it by a long upward arrow ↑ and put it on the right of the BYT. In [15, 16], they represent it by a column of 5 ⌟'s which means "half" of a column of 5 boxes (as $[0,0,0,0,1]$ is "half" of $[0,0,0,0,2]$ in terms of Dynkin labels), and put it on the right of the BYT.

spinorial irreps. The alternate coloring was just to illustrate the origin of those digits. Two simple extra examples are

(5.3)
$$\boxed{}\,\boxed{16}_{\text{IR}} \quad = [1,0,0,0,1] \quad ,$$

(5.4)
$$\text{(Young tableau)}_{\text{IR}} \quad = [0,1,1,1,0] \quad .$$

Note that although there is no ambiguity in translating from a mixed YT to a Dynkin Label for a spinorial irrep as in Equation (5.2), there is an ambiguity for the converse translation. For a general bosonic irrep, the Dynkin Label $[a, b, c, d, e]$ has 5 integers, while in the BYT there are 5 numbers p, q, r, s, and t to describe the numbers of columns with 1, 2, 3, 4, and 5 boxes respectively. So they already form a one-to-one mapping between Dynkin Labels and BYTs. Now the basic spinorial irreps in Equation (5.1) are introduced. New YT notations $\boxed{16}$ and $\boxed{\overline{16}}$ are introduced, so there are 7 fundamental objects for YTs. A general Dynkin Label $[a, b, c, d, e]$ remains to have only 5 integers, and there's one more degree of freedom that we can gain from the Dynkin Label: the parity of $|e - d|$. Therefore, we have 6 degrees of freedom for Dynkin Labels. So ambiguities arise. The simplest example is stated below.

(5.5)
$$[0,0,0,1,2] = [0,0,0,1,1] + [0,0,0,0,1] \quad ,$$
$$[0,0,0,1,2] = [0,0,0,0,2] + [0,0,0,1,0] \quad ,$$

where the "+" here just means viewing each Dynkin Label as a vector and summing them integer by integer, but *not* any sort of direct sum of irreps. The way to resolve this is to eliminate one degree of freedom from the spinorial YT notation by making a choice - for the following dimension rules, we interpret *every* spinorial irrep (an irrep is spinorial when $|e - d|$ is odd) to be $[0,0,0,0,1]$ "+" some other bosonic Dynkin Labels, when $e > d$. When $e < d$, perform the conjugation first, apply the dimension rule, and carry out the conjugation again to obtain the final result. Interpreted in another way, that would mean for $e < d$, it would be $[0,0,0,1,0]$ "+" some other bosonic Dynkin Labels. With this condition, for the example in Equation (5.5), we *pick*

(5.6)
$$[0,0,0,1,2] = [0,0,0,1,1] + [0,0,0,0,1] = \text{(Young tableau)}_{\text{IR}} \quad ,$$

while for $[0,0,0,2,1]$, since $e < d$, we carry out the conjugation first to find $[0,0,0,1,2]$, map it to YT notation according to Equation (5.6), then perform conjugation again. Therefore,

$$(5.7) \qquad [0,0,0,2,1] \; = \; [0,0,0,1,1] + [0,0,0,1,0] \; =$$

Now the ambiguity has been cleared. This also brings us to two remarks.

1) Since we choose to translate a Dynkin Label $[a,b,c,d,e]$ with $e > d$ to YT notation, so in mixed YT notation, only self-dual BYTs remain - but of course if we perform a conjugation we obtain the anti-self-dual parts. Therefore, a general spinorial irrep (or its conjugate) can be represented by

$$(5.8)$$

2) While writing the irrep in dimension notation, it is $\{\mathrm{dim}\}$ if the number of boxes in the BYT part is even, and $\{\overline{\mathrm{dim}}\}$ if odd.

For the following dimension rules, we would focus on the $e > d$ cases (and the readers can obtain the $e < d$ cases by conjugation). For convenience of presentation, from now on, ☐ represents $\boxed{16}$ unless specified otherwise. And the overall spinorial irrep would always be specified.

5.1. General spinorial irrep graphical dimension rules

How do we calculate the dimensions of these spinorial irreps graphically then? And what about their irreducible constraints? It turns out that these two questions are two sides of the same coin. In [36], the general formula for D_n algebra spinorial irrep was first given,

$$(5.9) \qquad\qquad [\Delta; \lambda] \; = \; [\lambda/P] \otimes \Delta \quad,$$

where Δ is the basic spinorial irrep, λ is a BYT shape, and

$$(5.10) \qquad P = \sum_m (-1)^m \overbrace{\boxed{\cdots}}^{m}{}_{\text{IR}} \quad ,$$

is a Schur function series.

A direct translation into our notation is

(5.11)

$$
\overbrace{}^{t}\overbrace{}^{s}\overbrace{}^{r}\overbrace{}^{q}\overbrace{}^{p}
$$

$$
= \sum_{m=0}^{p+q+r+s+t} (-1)^m \left(\left. \underset{\text{IR},+}{\underbrace{}^{\text{IR},+}} \middle/ \overbrace{\boxed{\cdots}}^{m}{}_{\text{IR}} \right) \otimes \boxed{\overline{16}}^{m} \right. \quad ,
$$

where on the right hand side

$$(5.12) \qquad \boxed{\overline{16}}^{m} = \begin{cases} \boxed{16} & , \quad \text{for } m \text{ even} \\ \boxed{\overline{16}} & , \quad \text{for } m \text{ odd} \end{cases} .$$

The SYT dimension formula is just taking the "dim" operator on both sides,

$$
\dim \left(\;\;\;\overbrace{}^{t}\overbrace{}^{s}\overbrace{}^{r}\overbrace{}^{q}\overbrace{}^{p}\;\;\; \right)_{\text{IR},+}
$$

$$(5.13) \quad = \left(\sum_{m=0}^{p+q+r+s+t} (-1)^m \dim \left(\overbrace{}^{t} \overbrace{}^{s} \overbrace{}^{r} \overbrace{}^{q} \overbrace{}^{p} \middle/ \overbrace{}^{m}_{\text{IR}} \right)_{\text{IR},+} \right)$$

$$\times \ \dim \left(\square \right) \ .$$

If we look at the right hand side of Equation (5.9), we notice that the dimension of the basic spinorial irrep is pulled out. Let us focus on the bosonic part $[\lambda/P]$, i.e. the $\sum_{m=0}^{p+q+r+s+t}$ part in Equation (5.13). If we look at it term-by-term, m is the number of box(es) in a row removed in each term, and it ranges from $m = 0$ (no box removed) to $m = p + q + r + s + t$ (entire row removed). Therefore, in the sum,

(5.14)

original diagram $\longrightarrow$ last diagram

This turns out to be an interesting combinatorics problem. Since the "/" operator would only count all the equivalent removals once, e.g.

(5.15)

the quotient operator for each m

(5.16)

can be thought as removing m objects from the following $p + q + r + s + t$ objects,

(5.17)

$$\overbrace{\boxed{t \cdots t}}^{t} \overbrace{\boxed{s \cdots s}}^{s} \overbrace{\boxed{r \cdots r}}^{r} \overbrace{\boxed{q \cdots q}}^{q} \overbrace{\boxed{p \cdots p}}^{p} \quad ,$$

with of course all p objects are indistinguishable, all q objects are indistinguishable, etc.. Therefore, for each m, the problem of how many terms appear there becomes:

> Given a multiset $\{p, \cdots, p, q, \cdots, q, r, \cdots, r, s, \cdots, s, t, \cdots, t\}$,
> how many subsets with a fixed size m are there?

Although we do not know the close form, we have an efficient algorithm to obtain the number[11]. If we write

(5.18) $(1 + x + \cdots + x^p)(1 + x + \cdots + x^q)(1 + x + \cdots + x^r)$

$$\times (1 + x + \cdots + x^s)(1 + x + \cdots + x^t) \ = \ \sum_{m=0}^{p+q+r+s+t} c_m x^m \quad ,$$

then c_m equals the number of subsets of size m. To obtain all the subsets, just modify the above polynomial to

(5.19) $(1 + p + \cdots + p^p)(1 + q + \cdots + q^q)$

$$\times (1 + r + \cdots + r^r)(1 + s + \cdots + s^s)(1 + t + \cdots + t^t) \quad ,$$

and take the part of the homogeneous polynomial of degree m in it. Each term then corresponds to a subset, or a combination.

To remove the abstractness, let us look at an example. Consider a mixed YT with the BYT part with $(p, q, r, s, t) = (1, 2, 1, 0, 0)$, i.e.

(5.20)

IR

and m ranges from 0 to 4 $(= 1 + 2 + 1)$. Application of Equations (5.18) and (5.19) gives us the combinations in Table 1.

---

[11] SNHM thanks Kevin Iga for this idea.

m	Number of combinations c_m	Combinations
0	1	1
1	3	p, q, r
2	4	pq, pr, q^2, qr
3	3	pq^2, pqr, q^2r
4	1	pq^2r

Table 1: All possible combinations of removing m boxes for $(p, q, r, s, t) = (1, 2, 1, 0, 0)$

Now let us translate back to the BYT notation and look at the terms in the sum one by one.

(5.21)
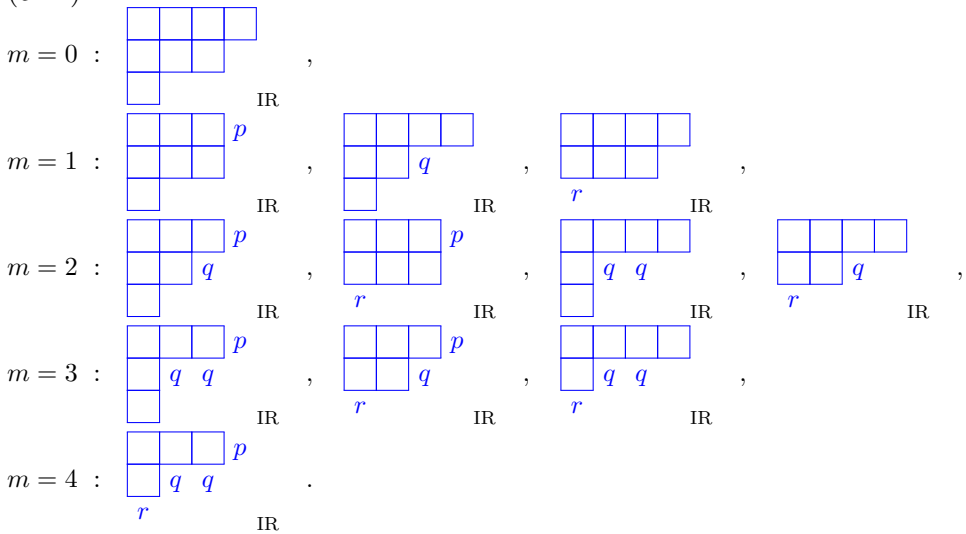
Another point to note is that each term is attached by a $(-1)^m$ factor. That means when even number of boxes are removed, we add the dimensions of those terms; when odd number of boxes are removed, we subtract their dimensions. Therefore,

(5.22)

$$
\begin{aligned}
&-\dim\left(\young{}\,\right)_{\text{IR}} -\dim\left(\young{}\,\right)_{\text{IR}} -\dim\left(\young{}\,\right)_{\text{IR}} \\
&+\dim\left(\young{}\,\right)_{\text{IR}} +\dim\left(\young{}\,\right)_{\text{IR}} +\dim\left(\young{}\,\right)_{\text{IR}} \\
&+\dim\left(\young{}\,\right)_{\text{IR}} -\dim\left(\young{}\,\right)_{\text{IR}} -\dim\left(\young{}\,\right)_{\text{IR}} \\
&-\dim\left(\young{}\,\right)_{\text{IR}} +\dim\left(\young{}\,\right)_{\text{IR}} \Big) \times \dim\left(\square\right) \quad .
\end{aligned}
$$

Numerically,

$$
\begin{aligned}
(5.23)\qquad 1260000 = \big(\ & 174636 \\
& - 34398 - 68640 - 37362 \\
& + 17920 + 7644 + 14784 + 16380 \\
& - 4312 - 4410 - 4608 \\
& + 1386\ \big) \times 16 \quad .
\end{aligned}
$$

In the following sections, we will apply the spinorial dimension rule (5.13) for the spinorial irreps composed of completely antisymmetric, completely symmetric, two-equal-column, and two-unequal-column BYTs attached with a $\boxed{16}$. These are the types of spinorial irreps that would appear in the 10D, $\mathcal{N}=1$ scalar superfield. The general dimension formulas and the irreducible conditions will be presented in Sections 5.2, 5.3, 5.4 and 5.5, while the explicit details of the corresponding examples will be presented in Appendix B.

5.2. Completely antisymmetric BYTs attached with {16}

First let us consider totally antisymmetric BYTs attached with a $\boxed{16}$. By applying (5.13), one has

$$
(5.24) \quad \dim\left(n \left\{ \begin{array}{c} \vdots \end{array} \right._{\text{IR}} \right)
$$

$$
= \left(\dim\left(n \left\{ \begin{array}{c} \vdots \end{array} \right._{\text{IR}} \right) - \dim\left((n-1) \left\{ \begin{array}{c} \vdots \end{array} \right._{\text{IR}} \right) \right)
$$

$$
\times \dim\left(\square \right) .
$$

This formula for calculating the irrep dimensions (or the degrees of freedom of a field) is very suggestive about how we should write out the irreducible conditions. First note that the removal of a box from the BYT part in the mixed YT is equivalent to the contraction by a $(\sigma^{\underline{a}})_{\alpha\beta}$ matrix. If we use the index notation we invented in Section 3.3, with the general field on the left and the general irreducible condition on the right, we find

$$
(5.25) \qquad \Psi_{\{\underline{a}_1\cdots\underline{a}_n\}}{}^{\alpha} \quad : \qquad (\sigma^{\underline{a}_n})^{\alpha\beta}\Psi_{\{\underline{a}_1\cdots\underline{a}_n\}\beta} = 0 \quad ,
$$

where $n = 1, \ldots, 5$. The degree of freedom of the irreducible condition is that of d.o.f.$(\{\underline{a}_1\cdots\underline{a}_{n-1}\}) \times$ d.o.f.(α) as the sigma matrix contracted one vector index out. Therefore, it is consistent with the dimension formula as written in Equation (5.24), or in index notation,

$$
(5.26) \qquad \dim\left(\Psi_{\{\underline{a}_1\cdots\underline{a}_n\}}{}^{\alpha}\right) = \text{d.o.f.}(\{\underline{a}_1\cdots\underline{a}_n\}) \times \text{d.o.f.}(\alpha)
$$
$$
- \text{d.o.f.}(\{\underline{a}_1\cdots\underline{a}_{n-1}\}) \times \text{d.o.f.}(\alpha) .
$$

5.3. Completely symmetric BYTs attached with {16}

Now let us turn to totally symmetric BYTs attached with a $\boxed{16}$.

$$
(5.27) \quad \dim\left(\underbrace{\square \cdots \square}_{n} {}_{\text{IR}} \right) = \left(\dim\left(\underbrace{\square \cdots \square}_{n} {}_{\text{IR}} \right) \right.
$$
$$
\left. - \dim\left(\underbrace{\square \cdots \square}_{n-1} {}_{\text{IR}} \right) + \cdots \right.
$$

$$+ (-1)^{n-1} \dim \left(\boxed{}_{\text{IR}} \right) + (-1)^n \dim \left(\cdot \right) \right) \times \dim \left(\boxed{} \right) .$$

Again, we can write the analytical expressions of the fields on the left and the corresponding set of irreducible constraints on the right as follows,

(5.28)

$$\Psi_{\{\underline{a}_1,\cdots,\underline{a}_n\}}{}^\alpha : \begin{cases} (\sigma^{\underline{a}_n})_{\alpha\beta}\Psi_{\{\underline{a}_1,\cdots,\underline{a}_n\}}{}^\alpha \equiv \psi_{\{\underline{a}_1,\cdots,\underline{a}_{n-1}\}\beta} = 0 , \\ (\sigma^{\underline{a}_{n-1}})^{\gamma\beta}\psi_{\{\underline{a}_1,\cdots,\underline{a}_{n-1}\}\beta} \equiv \psi_{\{\underline{a}_1,\cdots,\underline{a}_{n-2}\}}{}^\gamma = 0 , \\ \qquad\qquad \vdots \\ (\sigma^{\underline{a}_1})^{\gamma\beta}\psi_{\{\underline{a}_1\}\beta} = 0 \ (n \text{ odd}) , \ (\sigma^{\underline{a}_1})_{\gamma\beta}\psi_{\{\underline{a}_1\}}{}^\beta = 0 \ (n \text{ even}) , \end{cases}$$

where $n = 1, 2, \ldots$. From these conditions, the dimension formula would be

(5.29)
$$\begin{aligned} \dim \left(\Psi_{\{\underline{a}_1,\cdots,\underline{a}_n\}}{}^\alpha \right) &= \text{d.o.f.}(\{\underline{a}_1,\cdots,\underline{a}_n\}) \times \text{d.o.f.}(\alpha) \\ &\quad - \Big(\text{d.o.f.}(\{\underline{a}_1,\cdots,\underline{a}_{n-1}\}) \times \text{d.o.f.}(\alpha) \\ &\quad\quad - \big(\text{d.o.f.}(\{\underline{a}_1,\cdots,\underline{a}_{n-2}\}) \times \text{d.o.f.}(\alpha) - \cdots \\ &\quad\quad\quad - \big(\text{d.o.f.}(\{\underline{a}_1\}) \times \text{d.o.f.}(\alpha) - \text{d.o.f.}(\alpha) \big) \big) \Big) \\ &= \Big(\text{d.o.f.}(\{\underline{a}_1,\cdots,\underline{a}_n\}) - \text{d.o.f.}(\{\underline{a}_1,\cdots,\underline{a}_{n-1}\}) \\ &\quad + \text{d.o.f.}(\{\underline{a}_1,\cdots,\underline{a}_{n-2}\}) \\ &\quad - \cdots + (-1)^{n-1}\text{d.o.f.}(\{\underline{a}_1\}) + (-1)^n \Big) \times \text{d.o.f.}(\alpha) , \end{aligned}$$

which agrees with Equation (5.27). One can see that for $n = 1$, it recovers the $n = 1$ case of the totally antisymmetric BYT attached with a $\boxed{16}$ in Equation (5.26).

5.4. Two-equal-column BYTs attached with {16}

How about two column BYTs with same number of boxes in each column, attached with a $\boxed{16}$? We have

(5.30)

$$
\dim\left(\;\overset{}{\underset{\text{IR}}{\vphantom{a}}}\;\right)
$$

$$
=\left(\;\dim\left(\;\overset{}{\underset{\text{IR}}{\vphantom{a}}}\;\right)\;-\;\dim\left(\;\overset{}{\underset{\text{IR}}{\vphantom{a}}}\;\right)\;+\;\dim\left(\;\overset{}{\underset{\text{IR}}{\vphantom{a}}}\;\right)\right)
$$

$$
\times\;\dim\left(\;\square\;\right)\;.
$$

The general form of the irreps with index notations on the left and the set of irreducible constraints on the right is

(5.31)

$$
\Psi_{\{\underline{a}_1\cdots\underline{a}_n,\underline{b}_1\cdots\underline{b}_n\}}{}^{\alpha}\;:\;\begin{cases}(\sigma^{\underline{b}_n})_{\alpha\beta}\Psi_{\{\underline{a}_1\cdots\underline{a}_n,\underline{b}_1\cdots\underline{b}_n\}}{}^{\alpha}\;\equiv\;\psi_{\{\underline{a}_1\cdots\underline{a}_n|\underline{b}_1\cdots\underline{b}_{n-1}\}\beta}\;=\;0\;,\\(\sigma^{\underline{a}_n})^{\gamma\beta}\psi_{\{\underline{a}_1\cdots\underline{a}_n|\underline{b}_1\cdots\underline{b}_{n-1}\}\beta}\;=\;0\;,\end{cases}
$$

where $n=1,\ldots,5$. The dimension formula derived from these constraints is

$$
\begin{aligned}
&\dim\left(\Psi_{\{\underline{a}_1\cdots\underline{a}_n,\underline{b}_1\cdots\underline{b}_n\}}{}^{\alpha}\right)\\
&\quad=\;\text{d.o.f.}(\{\underline{a}_1\cdots\underline{a}_n,\underline{b}_1\cdots\underline{b}_n\})\times\text{d.o.f.}(\alpha)\\
&\qquad-\Big(\;\text{d.o.f.}(\{\underline{a}_1\cdots\underline{a}_n|\underline{b}_1\cdots\underline{b}_{n-1}\})\times\text{d.o.f.}(\alpha)\\
&\qquad-\text{d.o.f.}(\{\underline{a}_1\cdots\underline{a}_{n-1},\underline{b}_1\cdots\underline{b}_{n-1}\})\times\text{d.o.f.}(\alpha)\;\Big)\\
&\quad=\Big(\;\text{d.o.f.}(\{\underline{a}_1\cdots\underline{a}_n,\underline{b}_1\cdots\underline{b}_n\})\;-\;\text{d.o.f.}(\{\underline{a}_1\cdots\underline{a}_n|\underline{b}_1\cdots\underline{b}_{n-1}\})\\
&\qquad+\text{d.o.f.}(\{\underline{a}_1\cdots\underline{a}_{n-1},\underline{b}_1\cdots\underline{b}_{n-1}\})\;\Big)\times\text{d.o.f.}(\alpha)\;.
\end{aligned}
$$

(5.32)

which corresponds to Equation (5.30). For $n=1$, this formula reduces to the dimension formula for the $n=2$ case of the totally symmetric BYT attached with a $\boxed{16}$ in Equation (5.29).

5.5. Two-unequal-column BYTs attached with $\mathbf{\{16\}}$

Last but not least, one may wonder about the dimensions of the spinorial irreps represented by two-column BYTs with different number of boxes in

the two columns attached with a $\boxed{16}$,

(5.33)

$$\mathrm{dim}\left(\vcenter{\hbox{\includegraphics{}}}_{\mathrm{IR}}\right) = \left(\mathrm{dim}\left(\vcenter{\hbox{\includegraphics{}}}_{\mathrm{IR}}\right) - \mathrm{dim}\left(\vcenter{\hbox{\includegraphics{}}}_{\mathrm{IR}}\right)\right.$$

$$\left. - \mathrm{dim}\left(\vcenter{\hbox{\includegraphics{}}}_{\mathrm{IR}}\right) + \mathrm{dim}\left(\vcenter{\hbox{\includegraphics{}}}_{\mathrm{IR}}\right)\right)$$

$$\times \ \mathrm{dim}\left(\square\right) \ .$$

Now let us give the general expression for the fields and the general set of irreducible conditions. There are two equivalent ways to write the set of irreducible conditions.

$$\Psi_{\{\underline{a}_1\cdots\underline{a}_n|\underline{b}_1\cdots\underline{b}_m\}}{}^\alpha \ \ (m > n) \ \ :$$

(5.34)

$$\begin{cases} (\sigma^{\underline{b}_m})_{\alpha\beta}\Psi_{\{\underline{a}_1\cdots\underline{a}_n|\underline{b}_1\cdots\underline{b}_m\}}{}^\alpha \equiv \psi_{\{\underline{a}_1\cdots\underline{a}_n|\underline{b}_1\cdots\underline{b}_{m-1}\}\beta} = 0 \ , \\ (\sigma^{\underline{a}_n})_{\alpha\beta}\Psi_{\{\underline{a}_1\cdots\underline{a}_n|\underline{b}_1\cdots\underline{b}_m\}}{}^\alpha = 0 \ , \\ (\sigma^{\underline{a}_n})^{\gamma\beta}\psi_{\{\underline{a}_1\cdots\underline{a}_n|\underline{b}_1\cdots\underline{b}_{m-1}\}\beta} = 0 \ , \end{cases}$$

or

$$\begin{cases} (\sigma^{\underline{b}_m})_{\alpha\beta}\Psi_{\{\underline{a}_1\cdots\underline{a}_n|\underline{b}_1\cdots\underline{b}_m\}}{}^\alpha = 0 \ , \\ (\sigma^{\underline{a}_n})_{\alpha\beta}\Psi_{\{\underline{a}_1\cdots\underline{a}_n|\underline{b}_1\cdots\underline{b}_m\}}{}^\alpha \equiv \psi_{\{\underline{a}_1\cdots\underline{a}_{n-1}|\underline{b}_1\cdots\underline{b}_m\}\beta} = 0 \ , \\ (\sigma^{\underline{b}_m})^{\gamma\beta}\psi_{\{\underline{a}_1\cdots\underline{a}_{n-1}|\underline{b}_1\cdots\underline{b}_m\}\beta} = 0 \ , \end{cases}$$

where $m = 2, \ldots, 5$ and $n = 1, \ldots, m - 1$. The general dimension formula is

(5.35)
$$
\begin{aligned}
\dim &\left(\Psi_{\{\underline{a}_1 \cdots \underline{a}_n | \underline{b}_1 \cdots \underline{b}_m\}}{}^\alpha \right) \\
= \; & \text{d.o.f.}(\{\underline{a}_1 \cdots \underline{a}_n | \underline{b}_1 \cdots \underline{b}_m\}) \times \text{d.o.f.}(\alpha) \\
& - \Big(\text{d.o.f.}(\{\underline{a}_1 \cdots \underline{a}_n | \underline{b}_1 \cdots \underline{b}_{m-1}\}) \times \text{d.o.f.}(\alpha) \\
& \quad - \text{d.o.f.}(\{\underline{a}_1 \cdots \underline{a}_{n-1} | \underline{b}_1 \cdots \underline{b}_{m-1}\}) \times \text{d.o.f.}(\alpha) \Big) \\
& - \text{d.o.f.}(\{\underline{a}_1 \cdots \underline{a}_{n-1} | \underline{b}_1 \cdots \underline{b}_m\}) \times \text{d.o.f.}(\alpha) \\
= \; & \text{d.o.f.}(\{\underline{a}_1 \cdots \underline{a}_n | \underline{b}_1 \cdots \underline{b}_m\}) \times \text{d.o.f.}(\alpha) \\
& - \Big(\text{d.o.f.}(\{\underline{a}_1 \cdots \underline{a}_{n-1} | \underline{b}_1 \cdots \underline{b}_m\}) \times \text{d.o.f.}(\alpha) \\
& \quad - \text{d.o.f.}(\{\underline{a}_1 \cdots \underline{a}_{n-1} | \underline{b}_1 \cdots \underline{b}_{m-1}\}) \times \text{d.o.f.}(\alpha) \Big) \\
& - \text{d.o.f.}(\{\underline{a}_1 \cdots \underline{a}_n | \underline{b}_1 \cdots \underline{b}_{m-1}\}) \times \text{d.o.f.}(\alpha) \\
= \; & \Big(\text{d.o.f.}(\{\underline{a}_1 \cdots \underline{a}_n | \underline{b}_1 \cdots \underline{b}_m\}) - \text{d.o.f.}(\{\underline{a}_1 \cdots \underline{a}_n | \underline{b}_1 \cdots \underline{b}_{m-1}\}) \\
& \quad - \text{d.o.f.}(\{\underline{a}_1 \cdots \underline{a}_{n-1} | \underline{b}_1 \cdots \underline{b}_m\}) \\
& \quad + \text{d.o.f.}(\{\underline{a}_1 \cdots \underline{a}_{n-1} | \underline{b}_1 \cdots \underline{b}_{m-1}\}) \Big) \times \text{d.o.f.}(\alpha) \; ,
\end{aligned}
$$

and one can check that it agrees with Equation (5.33). When one compares it to the dimension formula for two-equal-column BYT attached with a $\boxed{16}$ in Equation (5.32), one quickly understands the relation between that formula and this dimension formula (5.35). Since for the case of two-equal-column mixed YT, $m = n$ and $\psi_{\{\underline{a}_1 \cdots \underline{a}_n | \underline{b}_1 \cdots \underline{b}_{m-1}\}\beta} = \psi_{\{\underline{a}_1 \cdots \underline{a}_{n-1} | \underline{b}_1 \cdots \underline{b}_m\}\beta}$, so it was not counted twice and there was only one term d.o.f.$(\{\underline{a}_1 \cdots \underline{a}_n | \underline{b}_1 \cdots \underline{b}_{n-1}\})$ in Equation (5.32). For Equation (5.35), however, we have both d.o.f.$(\{\underline{a}_1 \cdots \underline{a}_n | \underline{b}_1 \cdots \underline{b}_{m-1}\})$ and d.o.f.$(\{\underline{a}_1 \cdots \underline{a}_{n-1} | \underline{b}_1 \cdots \underline{b}_m\})$ terms. This exactly reflects the meaning of "/" operator of BYT - that equivalent removals of one box will be only counted once, as indicated in Equation (5.15).

6. Tensor product rules of a bosonic irrep with the basic spinorial irrep

For a D_n algebra, the tensor product rule of a general BYT $[\lambda]$ with the basic spinorial irrep $\Delta = \{16\}$ was first given by Murnaghan and Littlewood in

1938 and 1950 respectively [32, 34],

$$(6.1) \qquad\qquad [\lambda] \otimes \Delta \;=\; [\Delta; \lambda/Q] \quad,$$

where Q is a Schur function series defined by

$$(6.2) \qquad\qquad Q \;=\; \sum_n \;\left.\begin{array}{c}\vdots\end{array}\right\}_{\mathrm{IR}} n \quad.$$

It turns out that the spinorial irrep dimension rule in Equation (5.9) was derived from this tensor product rule. In [36], King found the "inverse" of the Schur function series Q to be P, as explicitly defined in Equation (5.10).

Following the footsteps of Section 5.1, we translate the tensor product rule to our notation.

(6.3)

where $\boxed{\overline{16}}^{\,n}$ is attached to the BYT after the quotient operation; and

$$(6.4) \qquad h \;=\; \begin{cases} 5 & , \quad \text{for } t \neq 0 \\ 4 & , \quad \text{for } t = 0 \text{ and } s \neq 0 \\ 3 & , \quad \text{for } t = s = 0 \text{ and } r \neq 0 \\ 2 & , \quad \text{for } t = s = r = 0 \text{ and } q \neq 0 \\ 1 & , \quad \text{for } t = s = r = q = 0 \text{ and } p \neq 0 \\ 0 & , \quad \text{for } t = s = r = q = p = 0 \end{cases} \;,$$

is the height of the BYT. The tensor product rule is very similar to the SYT dimension rule in terms of combinatorics, and it's even simpler as we do not

have the alternating signs in the final sum - we just take the direct sum of all possible objects we obtain.

Let us show an example to illustrate the idea,

$$(6.5) \qquad \text{IR} \otimes \square \ .$$

Since we have three rows of different numbers of boxes, we consider the three rows to be different. So n ranges from 0 to $h = 3$, and what we do is to remove n objects from the following 3 objects,

$$(6.6) \qquad \boxed{\begin{matrix} L \\ M \\ N \end{matrix}} \ .$$

As these objects are all distinguishable, the combinatorics become very simple. The numbers of combinations are just the binomial coefficients $\binom{3}{n}$. If we look at the BYTs in the right hand side of Equation (6.3) without attaching the spinorial YT, term by term, we have

$$(6.7)$$

$n = 0$:

$n = 1$:

$n = 2$:

$n = 3$:

Therefore,

(6.8)

Or in dimension notation,

(6.9)
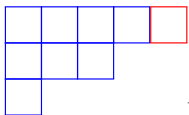
$$\{174636\} \otimes \{16\} = \{1260000\}$$
$$\oplus \{258720\} \oplus \{529200\} \oplus \{332640\}$$
$$\oplus \{144144\} \oplus \{70560\} \oplus \{155232\}$$
$$\oplus \{43680\} \quad .$$

To see how Schur function series P and Q as defined in Equations (5.10) and (6.2) are inverses of each other, let us use the above example to demonstrate the consistency of the SYT dimension rule (5.9) and the tensor product formula (6.1), i.e.

(6.10)

In Equation (5.22), the dimension formula for was shown.

Next, we could write down the dimension formulas for each term on the right hand side of (6.8). Note that the conjugation of an irrep would not change its dimension, so the same dimension formula applies. The dimension formula

can be written in the notation (with even more details left out)

$$(6.11) \qquad \dim([\Delta; \lambda]) \; = \; \left(\sum_i b_i \dim(\lambda_i) \right) \; \times \; \dim(\Delta) \qquad .$$

where b_i can take the values $+1$, -1 or 0. Then we can conveniently put all the relevant dimension formulas in Table 2.

	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	λ_{10}	λ_{11}	λ_{12}	λ_{13}	λ_{14}	λ_{15}	λ_{16}	λ_{17}	λ_{18}	λ_{19}
	1	-1	-1	-1	1	1	1	1	0	-1	-1	-1	0	0	1	0	0	0	0
	0	1	0	0	-1	-1	0	0	0	1	1	0	0	0	-1	0	0	0	0
	0	0	1	0	-1	0	-1	-1	1	1	1	1	-1	-1	-1	0	1	0	0
	0	0	0	1	0	-1	0	-1	0	0	1	1	0	0	-1	-1	0	1	0
	0	0	0	0	1	0	0	0	-1	-1	-1	0	1	1	1	0	-1	0	0
	0	0	0	0	0	1	0	0	0	0	-1	0	0	0	1	0	0	-1	0
	0	0	0	0	0	0	0	1	0	0	-1	-1	0	1	1	1	-1	-1	1
	0	0	0	0	0	0	0	0	0	0	1	0	0	-1	-1	0	1	1	-1
	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 2: Consistency check of the tensor product formula with the dimension formula and the demonstration of P as an inverse of Q

The first column contains all the terms on the right hand side of (6.8); and the first row contains all possible λ_i's in Equation (6.11) generated by them. The numbers in the middle of the table are b_i's. Each row of b_i's represents all the terms in the dimension formula for that irrep $[\Delta; \lambda]$ in the first entry of the row. By the tensor product results in Equation (6.8), adding up all the dimensions of the terms on the right hand side gives us the dimension of the tensor product. This operation corresponds to adding up each column of b_i's in the middle of the table, which gives us the last row of the table - the dimension for the tensor product. As we can see, only the leading term remains, and all other terms sum up to zero, i.e. they cancel with each other. Therefore, we verify Equation (6.10).

Moreover, we easily see that when summing up the terms in the Schur function series Q, the alternating signs generated by P naturally allow non-leading terms cancel with each other. It is in the sense that the Schur function series P is an inverse of the Schur function series Q.

Some more examples are shown in Appendix C, where we start from tensor product rules and then derive SYT dimension formulas. This approach is the converse of what we have just done.

7. Complete descriptions of component fields in 10D, $\mathcal{N} = 1$ scalar superfield

In this chapter, we present the complete descriptions of all component fields that occur in the 10D, $\mathcal{N} = 1$ scalar superfield decomposition. In [3] (and Section 2 above) we presented the Lorentz descriptions of all the component fields. By using the techniques discussed in the previous sections, we can translate the Dynkin Labels of the component fields in Figure 1 to Young Tableaux.

Or we can directly translate the whole adynkra into Figure 7.
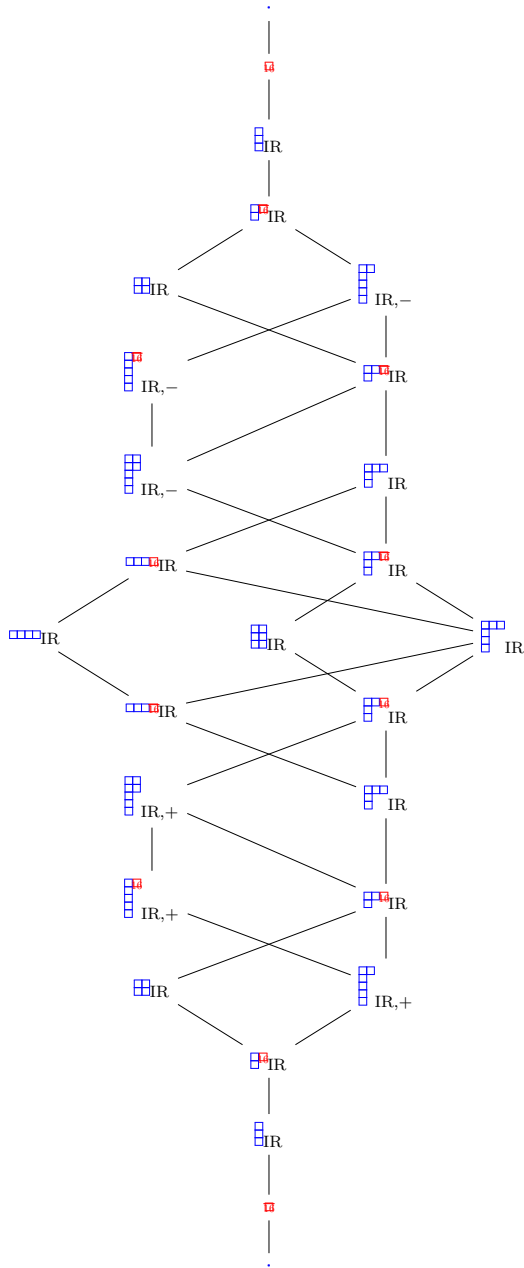
Figure 4: 10D, $\mathcal{N} = 1$ Adinkra in Young Tableaux

Given the data above, we can define a series $\widetilde{\mathcal{G}}(\ell)$ dependent on a real parameter "ℓ" via the equation
(7.2)

$$\widetilde{\mathcal{G}} = \cdot \ \oplus \ \ell\,\boxed{16} \ \oplus \ \tfrac{1}{2}\,(\ell)^2\,\square_{\mathrm{IR}} \ \oplus \ \tfrac{1}{3!}\,(\ell)^3\,\square\boxed{16}_{\mathrm{IR}} \ \oplus \ \tfrac{1}{4!}\,(\ell)^4\,\square_{\mathrm{IR}} \ \oplus \ \tfrac{1}{4!}\,(\ell)^4\,\square_{\mathrm{IR},-}$$

$$\oplus \ \tfrac{1}{5!}\,(\ell)^5\,\boxed{16}\square_{\mathrm{IR},-} \ \oplus \ \tfrac{1}{5!}\,(\ell)^5\,\square\boxed{16}_{\mathrm{IR}} \ \oplus \ \tfrac{1}{6!}\,(\ell)^6\,\square_{\mathrm{IR},-} \ \oplus \ \tfrac{1}{6!}\,(h)^6\,\square_{\mathrm{IR}}$$

$$\oplus \ \tfrac{1}{7!}\,(\ell)^7\,\square\boxed{16}_{\mathrm{IR}} \ \oplus \ \tfrac{1}{7!}\,(\ell)^7\,\square\boxed{16}_{\mathrm{IR}}$$

$$\oplus \ \tfrac{1}{8!}\,(\ell)^8\,\square\square\boxed{16}_{\mathrm{IR}} \ \oplus \ \tfrac{1}{8!}\,(\ell)^8\,\square_{\mathrm{IR}} \ \oplus \ \tfrac{1}{8!}\,(\ell)^8\,\square_{\mathrm{IR}}$$

$$\oplus \ \tfrac{1}{9!}\,(\ell)^9\,\square\boxed{16}_{\mathrm{IR}} \ \oplus \ \tfrac{1}{9!}\,(\ell)^9\,\boxed{16}\square_{\mathrm{IR}}$$

$$\oplus \ \tfrac{1}{10!}\,(\ell)^{10}\,\square_{\mathrm{IR},+} \ \oplus \ \tfrac{1}{10!}\,(\ell)^{10}\,\square_{\mathrm{IR}} \ \oplus \ \tfrac{1}{11!}\,(\ell)^{11}\,\boxed{16}\square_{\mathrm{IR},+}$$

$$\oplus \ \tfrac{1}{11!}\,(\ell)^{11}\,\square\boxed{16}_{\mathrm{IR}} \ \oplus \ \tfrac{1}{12!}\,(\ell)^{12}\,\square_{\mathrm{IR}} \ \oplus \ \tfrac{1}{12!}\,(\ell)^{12}\,\square_{\mathrm{IR},+}$$

$$\oplus \ \tfrac{1}{13!}\,(\ell)^{13}\,\boxed{16}\square_{\mathrm{IR}} \ \oplus \ \tfrac{1}{14!}\,(\ell)^{14}\,\square_{\mathrm{IR}} \ \oplus \ \tfrac{1}{15!}\,(\ell)^{15}\,\boxed{16} \ \oplus \ \tfrac{1}{16!}\,(\ell)^{16}\cdot \ \ .$$

We use the parameter "ℓ" to track the level of the YT's that follow it, which will play an important role when we define the product of two superfields. Since the maximal number of the level is sixteen, the product of two YT's contributes to $(\ell)^n$ and when $n > 16$ this piece will be ruled out. If we set $\ell = 1$, $\widetilde{\mathcal{G}}$ reduces to $\mathcal{G}$, which will be defined in (8.1) in the next Chapter. They are alternate forms to each other.

The index structures of all fifteen bosonic and twelve fermionic fields are identified below along with the level at which the fields occur in the adinkra

of the superfield.

(7.3)

$$
\mathcal{V} = \begin{cases}
\text{Level} - 0 & \Phi(x) \quad, \\
\text{Level} - 1 & \Psi_\alpha(x) \quad, \\
\text{Level} - 2 & \Phi_{\{\underline{a}_1\underline{b}_1\underline{c}_1\}}(x) \quad, \\
\text{Level} - 3 & \Psi_{\{\underline{a}_1\underline{b}_1\}}{}^\alpha(x) \quad, \\
\text{Level} - 4 & \Phi_{\{\underline{a}_1\underline{b}_1,\underline{a}_2\underline{b}_2\}}(x) \quad, \quad \Phi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^+}(x) \quad, \\
\text{Level} - 5 & \Psi_{\{\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^+}{}^\alpha(x) \quad, \quad \Psi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\}}{}^\alpha(x) \quad, \\
\text{Level} - 6 & \Phi_{\{\underline{a}_2\underline{b}_2|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^+}(x) \quad, \quad \Phi_{\{\underline{a}_2,\underline{a}_3|\underline{a}_1\underline{b}_1\underline{c}_1\}}(x) \quad, \\
\text{Level} - 7 & \Psi_{\{\underline{a}_1,\underline{a}_2,\underline{a}_3\}\alpha}(x) \quad, \quad \Psi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\underline{c}_1\}}{}^\alpha(x) \quad, \\
\text{Level} - 8 & \Phi_{\{\underline{a}_1,\underline{a}_2,\underline{a}_3,\underline{a}_4\}}(x) \quad, \quad \Phi_{\{\underline{a}_1\underline{b}_1\underline{c}_1,\underline{a}_2\underline{b}_2\underline{c}_2\}}(x) \quad, \\
& \Phi_{\{\underline{a}_2,\underline{a}_3|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\}}(x) \quad, \\
\text{Level} - 9 & \Psi_{\{\underline{a}_1,\underline{a}_2,\underline{a}_3\}}{}^\alpha(x) \quad, \quad \Psi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\underline{c}_1\}\alpha}(x) \quad, \\
\text{Level} - 10 & \Phi_{\{\underline{a}_2\underline{b}_2|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^-}(x) \quad, \quad \widehat{\Phi}_{\{\underline{a}_2,\underline{a}_3|\underline{a}_1\underline{b}_1\underline{c}_1\}}(x) \quad, \\
\text{Level} - 11 & \Psi_{\{\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^-}{}^\alpha(x) \quad, \quad \Psi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\}\alpha}(x) \quad, \\
\text{Level} - 12 & \widehat{\Phi}_{\{\underline{a}_1\underline{b}_1,\underline{a}_2\underline{b}_2\}}(x) \quad, \quad \Phi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^-}(x) \quad, \\
\text{Level} - 13 & \Psi_{\{\underline{a}_1\underline{b}_1\}\alpha}(x) \quad, \\
\text{Level} - 14 & \widehat{\Phi}_{\{\underline{a}_1\underline{b}_1\underline{c}_1\}}(x) \quad, \\
\text{Level} - 15 & \Psi^\alpha(x) \quad, \\
\text{Level} - 16 & \widehat{\Phi}(x) \quad.
\end{cases}
$$

These component fields are subject to irreducibility conditions listed below.

- Level-3:

(7.4)
$$
\Psi_{\{\underline{a}_1\underline{b}_1\}}{}^\alpha(x) \quad : \quad (\sigma^{\underline{a}_1})_{\alpha\beta}\Psi_{\{\underline{a}_1\underline{b}_1\}}{}^\alpha(x) = 0 \quad.
$$

- Level-4:

(7.5)
$$
\Phi_{\{\underline{a}_1\underline{b}_1,\underline{a}_2\underline{b}_2\}}(x) \quad : \quad \begin{cases} \eta^{\underline{a}_1\underline{a}_2}\Phi_{\{\underline{a}_1\underline{b}_1,\underline{a}_2\underline{b}_2\}}(x) = 0 \quad, \\ \eta^{\underline{a}_1\underline{a}_2}\eta^{\underline{b}_1\underline{b}_2}\Phi_{\{\underline{a}_1\underline{b}_1,\underline{a}_2\underline{b}_2\}}(x) = 0 \quad. \end{cases}
$$

(7.6)

$$
\Phi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^+}(x) \quad :
$$
$$
\begin{cases} \eta^{\underline{a}_1\underline{a}_2}\Phi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^+}(x) = 0 \quad, \\ \Phi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^+}(x) = +\tfrac{1}{5!}\epsilon_{\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1}{}^{\underline{f}_1\underline{g}_1\underline{h}_1\underline{i}_1\underline{j}_1}\Phi_{\{\underline{a}_2|\underline{f}_1\underline{g}_1\underline{h}_1\underline{i}_1\underline{j}_1\}^+}(x) \quad. \end{cases}
$$

- Level-5:

(7.7)

$$
\Psi_{\{\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^+}{}^\alpha(x) \quad : \\
\begin{cases}
(\sigma^{\underline{e}_1})_{\beta\alpha}\Psi_{\{\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^+}{}^\alpha(x) \;=\; 0 \quad, \\
\Psi_{\{\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^+}{}^\alpha(x) \;=\; +\tfrac{1}{5!}\epsilon_{\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1}{}^{\underline{f}_1\underline{g}_1\underline{h}_1\underline{i}_1\underline{j}_1}\Psi_{\{\underline{f}_1\underline{g}_1\underline{h}_1\underline{i}_1\underline{j}_1\}^+}{}^\alpha(x) \quad.
\end{cases}
$$

$$
(7.8) \;\; \Psi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\}}{}^\alpha(x) \quad : \quad
\begin{cases}
(\sigma^{\underline{b}_1})_{\alpha\delta}\Psi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\}}{}^\alpha(x) \;\equiv\; \psi_{\{\underline{a}_1,\underline{a}_2\}\delta}(x) \;=\; 0 \quad, \\
(\sigma^{\underline{a}_2})_{\alpha\delta}\Psi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\}}{}^\alpha(x) \;\equiv\; \psi_{\{\underline{a}_1\underline{b}_1\}\delta}(x) \;=\; 0 \quad, \\
(\sigma^{\underline{b}_1})^{\delta\epsilon}\psi_{\{\underline{a}_1\underline{b}_1\}\delta}(x) \;=\; 0 \quad.
\end{cases}
$$

- Level-6:

(7.9)

$$
\Phi_{\{\underline{a}_2\underline{b}_2|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^+}(x) \quad : \\
\begin{cases}
\eta^{\underline{a}_1\underline{a}_2}\Phi_{\{\underline{a}_2\underline{b}_2|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^+}(x) = 0 \quad, \\
\eta^{\underline{a}_1\underline{a}_2}\eta^{\underline{b}_1\underline{b}_2}\Phi_{\{\underline{a}_2\underline{b}_2|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^+}(x) = 0 \quad, \\
\Phi_{\{\underline{a}_2\underline{b}_2|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^+}(x) = \tfrac{1}{5!}\epsilon_{\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1}{}^{\underline{f}_1\underline{g}_1\underline{h}_1\underline{i}_1\underline{j}_1}\Phi_{\{\underline{a}_2\underline{b}_2|\underline{f}_1\underline{g}_1\underline{h}_1\underline{i}_1\underline{j}_1\}^+}(x) \quad.
\end{cases}
$$

$$
(7.10) \qquad \Phi_{\{\underline{a}_2,\underline{a}_3|\underline{a}_1\underline{b}_1\underline{c}_1\}}(x) \quad : \quad
\begin{cases}
\eta^{\underline{a}_2\underline{a}_3}\Phi_{\{\underline{a}_2,\underline{a}_3|\underline{a}_1\underline{b}_1\underline{c}_1\}}(x) = 0 \quad, \\
\eta^{\underline{a}_1\underline{a}_3}\Phi_{\{\underline{a}_2,\underline{a}_3|\underline{a}_1\underline{b}_1\underline{c}_1\}}(x) = 0 \quad.
\end{cases}
$$

- Level-7:

(7.11)

$$
\Psi_{\{\underline{a}_1,\underline{a}_2,\underline{a}_3\}\alpha}(x) \quad : \quad
\begin{cases}
(\sigma^{\underline{a}_3})^{\alpha\delta}\Psi_{\{\underline{a}_1,\underline{a}_2,\underline{a}_3\}\alpha}(x) \;\equiv\; \psi_{\{\underline{a}_1,\underline{a}_2\}}{}^\delta(x) \;=\; 0 \quad, \\
(\sigma^{\underline{a}_2})_{\epsilon\delta}\psi_{\{\underline{a}_1,\underline{a}_2\}}{}^\delta(x) \;\equiv\; \psi_{\{\underline{a}_1\}\epsilon}(x) \;=\; 0 \quad, \\
(\sigma^{\underline{a}_1})^{\tau\epsilon}\psi_{\{\underline{a}_1\}\epsilon}(x) \;=\; 0 \quad.
\end{cases}
$$

(7.12)

$$
\Psi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\underline{c}_1\}}{}^\alpha(x) \quad : \quad
\begin{cases}
(\sigma^{\underline{a}_2})_{\alpha\delta}\Psi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\underline{c}_1\}}{}^\alpha(x) \;\equiv\; \psi_{\{\underline{a}_1\underline{b}_1\underline{c}_1\}\delta}(x) \;=\; 0 \quad, \\
(\sigma^{\underline{c}_1})_{\alpha\delta}\Psi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\underline{c}_1\}}{}^\alpha(x) \;\equiv\; \psi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\}\delta}(x) \;=\; 0 \quad, \\
(\sigma^{\underline{c}_1})^{\delta\epsilon}\psi_{\{\underline{a}_1\underline{b}_1\underline{c}_1\}\delta}(x) \;=\; 0 \quad.
\end{cases}
$$

- Level-8:

$$(7.13) \qquad \Phi_{\{\underline{a}_1,\underline{a}_2,\underline{a}_3,\underline{a}_4\}}(x) \quad : \quad \begin{cases} \eta^{\underline{a}_3\underline{a}_4}\Phi_{\{\underline{a}_1,\underline{a}_2,\underline{a}_3,\underline{a}_4\}}(x) = 0 \quad , \\ \eta^{\underline{a}_1\underline{a}_2}\eta^{\underline{a}_3\underline{a}_4}\Phi_{\{\underline{a}_1,\underline{a}_2,\underline{a}_3,\underline{a}_4\}}(x) = 0 \quad . \end{cases}$$

$$(7.14) \quad \Phi_{\{\underline{a}_1\underline{b}_1\underline{c}_1,\underline{a}_2\underline{b}_2\underline{c}_2\}}(x) \quad : \quad \begin{cases} \eta^{\underline{c}_1\underline{c}_2}\Phi_{\{\underline{a}_1\underline{b}_1\underline{c}_1,\underline{a}_2\underline{b}_2\underline{c}_2\}}(x) = 0 \quad , \\ \eta^{\underline{b}_1\underline{b}_2}\eta^{\underline{c}_1\underline{c}_2}\Phi_{\{\underline{a}_1\underline{b}_1\underline{c}_1,\underline{a}_2\underline{b}_2\underline{c}_2\}}(x) = 0 \quad , \\ \eta^{\underline{a}_1\underline{a}_2}\eta^{\underline{b}_1\underline{b}_2}\eta^{\underline{c}_1\underline{c}_2}\Phi_{\{\underline{a}_1\underline{b}_1\underline{c}_1,\underline{a}_2\underline{b}_2\underline{c}_2\}}(x) = 0 \quad . \end{cases}$$

$$(7.15) \qquad \Phi_{\{\underline{a}_2,\underline{a}_3|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\}}(x) \quad : \quad \begin{cases} \eta^{\underline{a}_2\underline{a}_3}\Phi_{\{\underline{a}_2,\underline{a}_3|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\}}(x) = 0 \quad , \\ \eta^{\underline{a}_1\underline{a}_3}\Phi_{\{\underline{a}_2,\underline{a}_3|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\}}(x) = 0 \quad . \end{cases}$$

- Level-9:

(7.16)

$$\Psi_{\{\underline{a}_1,\underline{a}_2,\underline{a}_3\}}{}^{\alpha}(x) \quad : \quad \begin{cases} (\sigma^{\underline{a}_3})_{\alpha\delta}\Psi_{\{\underline{a}_1,\underline{a}_2,\underline{a}_3\}}{}^{\alpha}(x) \equiv \psi_{\{\underline{a}_1,\underline{a}_2\}\delta}(x) = 0 \quad , \\ (\sigma^{\underline{a}_2})^{\epsilon\delta}\psi_{\{\underline{a}_1,\underline{a}_2\}\delta}(x) \equiv \psi_{\{\underline{a}_1\}}{}^{\epsilon}(x) = 0 \quad , \\ (\sigma^{\underline{a}_1})_{\tau\epsilon}\psi_{\{\underline{a}_1\}}{}^{\epsilon}(x) = 0 \quad . \end{cases}$$

(7.17)

$$\Psi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\underline{c}_1\}\alpha}(x) \quad : \quad \begin{cases} (\sigma^{\underline{a}_2})^{\alpha\delta}\Psi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\underline{c}_1\}\alpha}(x) \equiv \psi_{\{\underline{a}_1\underline{b}_1\underline{c}_1\}}{}^{\delta}(x) = 0 \quad , \\ (\sigma^{\underline{c}_1})^{\alpha\delta}\Psi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\underline{c}_1\}\alpha}(x) \equiv \psi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\}}{}^{\delta}(x) = 0 \quad , \\ (\sigma^{\underline{c}_1})_{\delta\epsilon}\psi_{\{\underline{a}_1\underline{b}_1\underline{c}_1\}}{}^{\delta}(x) = 0 \quad . \end{cases}$$

- Level-10:

(7.18)

$$\Phi_{\{\underline{a}_2\underline{b}_2|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^-}(x) \quad : $$
$$\begin{cases} \eta^{\underline{a}_1\underline{a}_2}\Phi_{\{\underline{a}_2\underline{b}_2|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^-}(x) = 0 \quad , \\ \eta^{\underline{a}_1\underline{a}_2}\eta^{\underline{b}_1\underline{b}_2}\Phi_{\{\underline{a}_2\underline{b}_2|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^-}(x) = 0 \quad , \\ \Phi_{\{\underline{a}_2\underline{b}_2|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^-}(x) = -\frac{1}{5!}\epsilon_{\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1}{}^{\underline{f}_1\underline{g}_1\underline{h}_1\underline{i}_1\underline{j}_1}\Phi_{\{\underline{a}_2\underline{b}_2|\underline{f}_1\underline{g}_1\underline{h}_1\underline{i}_1\underline{j}_1\}^-}(x) \quad . \end{cases}$$

$$(7.19) \qquad \widehat{\Phi}_{\{\underline{a}_2,\underline{a}_3|\underline{a}_1\underline{b}_1\underline{c}_1\}}(x) \quad : \quad \begin{cases} \eta^{\underline{a}_2\underline{a}_3}\widehat{\Phi}_{\{\underline{a}_2,\underline{a}_3|\underline{a}_1\underline{b}_1\underline{c}_1\}}(x) = 0 \quad , \\ \eta^{\underline{a}_1\underline{a}_3}\widehat{\Phi}_{\{\underline{a}_2,\underline{a}_3|\underline{a}_1\underline{b}_1\underline{c}_1\}}(x) = 0 \quad . \end{cases}$$

• Level-11:

(7.20)

$$\Psi_{\{\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^-\alpha}(x) \quad :$$
$$\begin{cases} (\sigma^{\underline{e}_1})^{\beta\alpha}\Psi_{\{\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^-\alpha}(x) = 0 \quad , \\ \Psi_{\{\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^-\alpha}(x) = -\frac{1}{5!}\epsilon_{\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1}{}^{\underline{f}_1\underline{g}_1\underline{h}_1\underline{i}_1\underline{j}_1}\Psi_{\{\underline{f}_1\underline{g}_1\underline{h}_1\underline{i}_1\underline{j}_1\}^-\alpha}(x) \quad . \end{cases}$$

(7.21)

$$\Psi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\}\alpha}(x) \quad : \quad \begin{cases} (\sigma^{\underline{b}_1})^{\alpha\delta}\Psi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\}\alpha}(x) \equiv \psi_{\{\underline{a}_1,\underline{a}_2\}}{}^\delta(x) = 0 \quad , \\ (\sigma^{\underline{a}_2})^{\alpha\delta}\Psi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\}\alpha}(x) \equiv \psi_{\{\underline{a}_1\underline{b}_1\}}{}^\delta(x) = 0 \quad , \\ (\sigma^{\underline{b}_1})_{\delta\epsilon}\psi_{\{\underline{a}_1\underline{b}_1\}}{}^\delta(x) = 0 \quad . \end{cases}$$

• Level-12:

(7.22) $\quad \widehat{\Phi}_{\{\underline{a}_1\underline{b}_1,\underline{a}_2\underline{b}_2\}}(x) \quad : \quad \begin{cases} \eta^{\underline{a}_1\underline{a}_2}\widehat{\Phi}_{\{\underline{a}_1\underline{b}_1,\underline{a}_2\underline{b}_2\}}(x) = 0 \quad , \\ \eta^{\underline{a}_1\underline{a}_2}\eta^{\underline{b}_1\underline{b}_2}\widehat{\widehat{\Phi}}_{\{\underline{a}_1\underline{b}_1,\underline{a}_2\underline{b}_2\}}(x) = 0 \quad . \end{cases}$

(7.23)

$$\Phi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^-}(x) \quad :$$
$$\begin{cases} \eta^{\underline{a}_1\underline{a}_2}\Phi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^-}(x) = 0 \quad , \\ \Phi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^-}(x) = -\frac{1}{5!}\epsilon_{\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1}{}^{\underline{f}_1\underline{g}_1\underline{h}_1\underline{i}_1\underline{j}_1}\Phi_{\{\underline{a}_2|\underline{f}_1\underline{g}_1\underline{h}_1\underline{i}_1\underline{j}_1\}^-}(x) \quad . \end{cases}$$

• Level-13:

(7.24) $\qquad \Psi_{\{\underline{a}_1\underline{b}_1\}\alpha}(x) \quad : \quad (\sigma^{\underline{a}_1})^{\alpha\beta}\Psi_{\{\underline{a}_1\underline{b}_1\}\alpha}(x) = 0 \quad .$

The important point to note is this presentation of all the component fields in the 10D, $\mathcal{N} = 1$ scalar superfield, with their index structures showing a complete set of vector and spinor indices, is achieved *without* ever introducing any σ-matrices.

Let us here consider the results in this chapter from a slightly different perspective. We begin from the space of fields that we denote by $\{\mathcal{F}\}$. This space naturally is bisected $\{\mathcal{F}\} = \{\mathcal{F}\}_b \oplus \{\mathcal{F}\}_f$ according whether a specific field is a bosonic or fermionic representation of the Lorentz algebra.

Thus a representation of $\{\mathcal{F}\}$ can be written in the form

$$
(7.25) \qquad \{\mathcal{F}\} = \{ \underset{\substack{\text{scalar}\\ \text{spin}-0}}{\phi(x)} \,, \; \underset{\substack{\text{photon}\\ \text{spin}-1}}{A_{\underline{a}}(x)} \,, \; \underset{\substack{\text{graviton}\\ \text{spin}-2}}{h_{\underline{ab}}(x)} \,, \dots \}
$$

$$
\oplus \{ \underset{\substack{\text{spinor}\\ \text{spin}-\frac{1}{2}}}{\lambda^{\alpha}(x)} \,, \; \underset{\substack{\text{spinor}\\ \text{spin}-\frac{1}{2}}}{\lambda_{\alpha}(x)} \,, \; \underset{\substack{\text{gravitino}\\ \text{spin}-\frac{3}{2}}}{\psi_{\underline{a}}{}^{\beta}(x)} \,, \; \underset{\substack{\text{gravitino}\\ \text{spin}-\frac{3}{2}}}{\psi_{\underline{a}\beta}(x)} \,, \dots \} \,,
$$

and the ellipses in the equations denote the fact these are infinite dimensional sets[12]. One point the discussion in this chapter makes clear is that the irreducible decomposition of the fields in $\{\mathcal{F}\}$ can be efficiently accomplished by the replacement of the various vector and spinor indices with Dynkin Label as precisely the same information is conveyed. Moreover, since the Dynkin Labels provide a specification of YT's, the latter can replace space-teime vector and spinor indices. When this is done, (7.25) is replaced by

$$
(7.26) \qquad \{\mathcal{F}\} = \{ \underset{\substack{\text{scalar}\\ \text{spin}-0}}{\phi(x)} \,, \; \underset{\substack{\text{photon}\\ \text{spin}-1}}{A_{\square}(x)} \,, \; \underset{\substack{\text{graviton}\\ \text{spin}-2}}{h_{\square\square}(x)} \,, \dots \}
$$

$$
\oplus \{ \underset{\substack{\text{spinor}\\ \text{spin}-\frac{1}{2}}}{\lambda_{\boxed{16}}(x)} \,, \; \underset{\substack{\text{spinor}\\ \text{spin}-\frac{1}{2}}}{\lambda_{\boxed{16}}(x)} \,, \; \underset{\substack{\text{gravitino}\\ \text{spin}-\frac{3}{2}}}{\psi_{\square\boxed{16}}(x)} \,, \; \underset{\substack{\text{gravitino}\\ \text{spin}-\frac{3}{2}}}{\psi_{\square\boxed{16}}(x)} \,, \dots \} \,.
$$

Finally, in view of the perspective of Wigner, who indicated that free elementary particles are irreducible representations of Poincaré group, one can rewrite (7.26) in the form

$$
(7.27) \qquad \{\mathcal{F}\} = \{ \underset{\substack{\text{scalar}\\ \text{spin}-0}}{\cdot \, (x)} \,, \; \underset{\substack{\text{photon}\\ \text{spin}-1}}{\square(x)} \,, \; \underset{\substack{\text{graviton}\\ \text{spin}-2}}{\square\square(x)} \,, \dots \}
$$

$$
\oplus \{ \underset{\substack{\text{spinor}\\ \text{spin}-\frac{1}{2}}}{\boxed{16}(x)} \,, \; \underset{\substack{\text{spinor}\\ \text{spin}-\frac{1}{2}}}{\boxed{16}(x)} \,, \; \underset{\substack{\text{gravitino}\\ \text{spin}-\frac{3}{2}}}{\square\boxed{16}(x)} \,, \; \underset{\substack{\text{gravitino}\\ \text{spin}-\frac{3}{2}}}{\square\boxed{16}(x)} \,, \dots \} \,,
$$

where space-time dependent YT's replace fields.

The supersymmetry transformation is a map that acts between the spaces $\{\mathcal{F}\}_b$ and $\{\mathcal{F}\}_f$. Physicists have long assumed this map is homotopic to the identity map and thus assume the existence of an infinitesimal

---

[12] Above, for convenience only, we have used the spin nomenclature of the field space in 4D to describe these fields.

operator δ_Q which depends on a parameter ϵ^α (also valued in the double cover) with the property

$$\text{(7.28)} \qquad\qquad \delta_Q(\epsilon^\alpha)\,\mathcal{F} \;=\; \{\widetilde{\mathcal{F}}\}_f \;\oplus\; \{\widetilde{\mathcal{F}}\}_b \quad .$$

The elements of $\{\widetilde{\mathcal{F}}\}_f$ are linear in ϵ^α and linear in the elements of $\{\mathcal{F}\}_f$ and may involve tensors that are invariant under the action of isometries of the metric of the d-dimensional manifold and can involve first derivatives. The elements of $\{\widetilde{\mathcal{F}}\}_b$ are linear in ϵ^α and linear in the elements of $\{\mathcal{F}\}_b$ and may involve tensors that are invariant under the action of isometries of the metric of the D-dimensional Lorentz algebra acting on the manifold and can involve first derivatives.

There are another set of infinitesimal variations, "translations" that can be defined on the space of fields. These are denoted by δ_P and depend on parameters $\xi^{\underline{a}}$ where these parameters are valued in the tangent space to the manifold

$$\text{(7.29)} \qquad\qquad \delta_P(\xi^{\underline{a}})\,\mathcal{F} \;=\; (\xi^{\underline{a}}\partial_{\underline{a}}\{\mathcal{F}\}_b) \;\oplus\; (\xi^{\underline{a}}\partial_{\underline{a}}\{\mathcal{F}\}_f) \quad .$$

A system which consists of a subset of all the fields in $\{\mathcal{F}\}$ is supersymmetric if the subset realizes the following equation

$$\text{(7.30)} \qquad\qquad \delta_Q(\epsilon_1^\alpha)\,\delta_Q(\epsilon_2^\beta) \;-\; \delta_Q(\epsilon_2^\alpha)\,\delta_Q(\epsilon_1^\beta) \;=\; \delta_P(\xi^{\underline{a}}) \quad ,$$

where $\xi^{\underline{a}} \;=\; i2 < \epsilon_1^\alpha(\gamma^{\underline{a}})_{\alpha\beta}\epsilon_2^\beta >$. Systems satisfying this condition are said to be "off-shell supersymmetric" or to possess "off-shell spacetime supersymmetry." The remarkable fact is that now decades after its first statement, the general *irreducible* solution to this problem is still *not* known.

What the physics community has quite effectively used is the fact that there is a related set of equations on the space of fields that is simpler to solve.

A hypersurface in field space may be defined by imposing some differential equations on the fields. For example, the scalar field might be harmonic, satisfying the condition that its d'Alembertian vanishes. In physics we call such a condition "an equation of motion" if it is derivable by the extremization of some function, typically denoted by S, that we call the action. Let us denote such equations of motion generically by the symbol ∂S. Most of the discussions in the physics literature involve representations such that

$$\text{(7.31)} \qquad\qquad \delta_Q(\epsilon_1^\alpha)\,\delta_Q(\epsilon_2^\beta) \;-\; \delta_Q(\epsilon_2^\alpha)\,\delta_Q(\epsilon_1^\beta) \;=\; \delta_P(\xi^{\underline{a}}) \;+\; \partial S \quad .$$

Any formulation that is equivalent to superfields implies $\partial S = 0$ up to gauge transformations.

8. From superfields to adynkrafields

The path we have travelled began with the adynkra in Figure 1 that suggested a reconceptualization of the superfield in terms of Young Tableaux which possess a well understood algebra. This replacement eliminates the need σ-matrices in the expansion of the superfield. As a consequence of branching rules, derivable from decoration of the Young Tableaux, Fierz identities are eliminated with a huge savings in terms of computational costs. On the basis of these, far more efficient algorithms for the explicit component-level examination of superfields are achievable... as we have shown in the present work.

The adynkra shown in Figure 1 can be expressed totally in a field-independent manner and purely in terms of group-theoretical constructs mathematically in terms of $\mathcal{G}$ with the definition

$$(8.1) \quad \mathcal{G} = 1 \ \oplus \ \ell \left\{ \left(\boxed{} \right) \times [a_1, b_1, c_1, d_1, e_1] \right\}$$

$$\oplus \bigoplus_{p=2}^{16} \frac{1}{p!} \, (\ell)^p \left\{ \left(\boxed{} \, (\wedge \boxed{})^{p-2} \wedge \boxed{} \right) \times [a_p, b_p, c_p, d_p, e_p] \right\} \quad ,$$

and where a number of definitions must be understood and these include:

(a.) $\boxed{}$ denotes the SYT introduced in (5.1),

(b.) the $\wedge$ product denotes the usual rule for multiplying two tableaux, but restricted so that only single column resultants are kept,

(c.) $[a_p, b_p, c_p, d_p, e_p]$ denotes a Dynkin Label for an irrep in $\mathfrak{so}(10)$ where the
quantities a_p, b_p, c_p, d_p, and e_p are a set of integers,

(d.) $\mathcal{A} \times [a_p, b_p, c_p, d_p, e_p] = [a_p, b_p, c_p, d_p, e_p]$ where $\mathcal{A}$ is a single column SYT
containing the irrep $[a_p, b_p, c_p, d_p, e_p]$ otherwise $\mathcal{A} \times [a_p, b_p, c_p, d_p, e_p] = 0$,

(e.) $\mathcal{A} \times [a_p, b_p, c_p, d_p, e_p] = \mathrm{m} \, [a_p, b_p, c_p, d_p, e_p]$ if instead $\mathcal{A}$ contains the representation
$[a_p, b_p, c_p, d_p, e_p]$ m-times, and finally

(f.) $\{\mathcal{A} \times [a_p, b_p, c_p, d_p, e_p]\}$ is a notation implying independent sums to be taken over

all possible values of a_p, b_p ,c_p ,d_p, and e_p.

The mathematical object $\mathcal{G}$, as illustrated in the alternate form shown in (7.2) for the case of 10D, $\mathcal{N} = 1$ superfields, is the fundamental one we have been studying in the works of [2, 3] and that has allowed unprecedented clarity and access to the component field structures of high dimensional superfield theories.

For arbitrary dimensions D, we can generalize the result of (8.1) to be of the form

$$(8.2) \qquad \mathcal{G} = 1 \ \oplus \ \ell \left\{ \left(\Box \right) \times [Dk_1] \right\}$$

$$\oplus \bigoplus_{p=2}^{d} \frac{1}{p!} (\ell)^p \left\{ \left(\Box \, (\wedge \Box)^{p-2} \wedge \Box \right) \times [Dk_p] \right\} \quad,$$

where $[Dk_1]$ and $[Dk_p]$ denote Dynkin Labels appropriate for the fields over the D-dimensional manifold and the quantity d is the dimensionality of the minimal spinor representation for the D-dimensional manifold.

We may call $\widetilde{\mathcal{G}}$ and $\mathcal{G}$ "adynkra series," which appears to be an adapted and specialized series expansion in terms of Young Tableaux.

> *All of these results strongly suggest adynkras are pointing in the direction of using series expansion in terms of YT's as a tool to gain the most fundamental mathematical understanding of this class of problems.*

The results of (7.2) can be combined with those in (7.3) - (7.24) to yield an "adynkrafield."

$$+ \; \frac{1}{6!} \, (\ell)^6 \; \boxed{YT}_{IR,-} \; \Phi_{\{\underline{a}_2\underline{b}_2|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^+}(x)$$

$$+ \; \frac{1}{6!} \, (\ell)^6 \; \boxed{YT}_{IR} \; \Phi_{\{\underline{a}_2,\underline{a}_3|\underline{a}_1\underline{b}_1\underline{c}_1\}}(x) \; + \; \frac{1}{7!} \, (\ell)^7 \; \boxed{\;\;\;16}_{IR} \; \Psi_{\{\underline{a}_1,\underline{a}_2,\underline{a}_3\}\alpha}(x)$$

$$+ \; \frac{1}{7!} \, (\ell)^7 \; \boxed{YT}_{IR} \; \Psi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\underline{c}_1\}}{}^\alpha(x) \; + \; \frac{1}{8!} \, (\ell)^8 \; \boxed{\;\;\;\;}_{IR} \; \Phi_{\{\underline{a}_1,\underline{a}_2,\underline{a}_3,\underline{a}_4\}}(x)$$

$$+ \; \frac{1}{8!} \, (\ell)^8 \; \boxed{YT}_{IR} \; \Phi_{\{\underline{a}_1\underline{b}_1\underline{c}_1,\underline{a}_2\underline{b}_2\underline{c}_2\}}(x) \; + \; \frac{1}{8!} \, (\ell)^8 \; \boxed{YT}_{IR} \; \Phi_{\{\underline{a}_2,\underline{a}_3|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\}}(x)$$

$$+ \; \frac{1}{9!} \, (\ell)^9 \; \boxed{\;\;\;16}_{IR} \; \Psi_{\{\underline{a}_1,\underline{a}_2,\underline{a}_3\}}{}^\alpha(x) \; + \; \frac{1}{9!} \, (\ell)^9 \; \boxed{YT}_{IR} \; \Psi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\underline{c}_1\}\alpha}(x)$$

$$+ \; \frac{1}{10!} \, (\ell)^{10} \; \boxed{YT}_{IR,+} \; \Phi_{\{\underline{a}_2\underline{b}_2|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^-}(x)$$

$$+ \; \frac{1}{10!} \, (\ell)^{10} \; \boxed{YT}_{IR} \; \widehat{\Phi}_{\{\underline{a}_2,\underline{a}_3|\underline{a}_1\underline{b}_1\underline{c}_1\}}(x)$$

$$+ \; \frac{1}{11!} \, (\ell)^{11} \; \boxed{YT}_{IR,+} \; \Psi_{\{\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^-\alpha}(x)$$

$$+ \; \frac{1}{11!} \, (\ell)^{11} \; \boxed{YT}_{IR} \; \Psi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\}\alpha}(x) \; + \; \frac{1}{12!} \, (\ell)^{12} \; \boxed{\;\;}_{IR} \; \widehat{\Phi}_{\{\underline{a}_1\underline{b}_1,\underline{a}_2\underline{b}_2\}}(x)$$

$$+ \; \frac{1}{12!} \, (\ell)^{12} \; \boxed{YT}_{IR,+} \; \Phi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^-}(x) \frac{1}{13!} \, (\ell)^{13} \; \boxed{16}_{IR} \; \Psi_{\{\underline{a}_1\underline{b}_1\}\alpha}(x)$$

$$+ \; \frac{1}{14!} \, (\ell)^{14} \; \boxed{YT}_{IR} \; \widehat{\Phi}_{\{\underline{a}_1\underline{b}_1\underline{c}_1\}}(x) \; + \; \frac{1}{15!} \, (\ell)^{15} \; \boxed{16} \Psi^\alpha(x)$$

(8.3)

$$+ \; \frac{1}{16!} \, (\ell)^{16} \; \widehat{\Phi}(x) \quad ,$$

and it is to understand that each index on every field has all of it indices "tied" to indices in the YT that precedes it. Bosonic indices are tied to blue boxes in the YT. Fermionic indices are tied to the red boxes. It should be noted that at most one red box appears for all terms. The result in (8.3) arises from taking (8.1) followed by the use of branching rules in the cases

of even numbers of red box to replace them by blue boxes. Finally we have evaluated the "inner product" $\mathcal{G} \cdot \{\mathcal{F}\}$.

Further with this definition, it seems reasonable to define a supercovariant derivative $\mathcal{D}$ via the formula
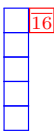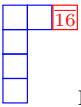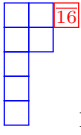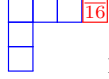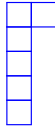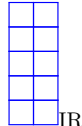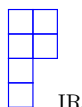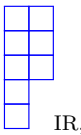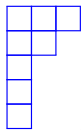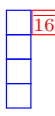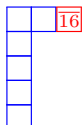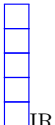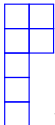
$$(8.4) \qquad \mathcal{D} \;=\; \partial_\ell \;+\; i\,\ell\;\boxed{\bullet}\!\!-\!\!\partial$$

when acting on such an adynkrafield. Here the ∂ operator is the derivative with respect to space-time coordinate. Its index is tied to the index in the blue box that precedes it. Continuing investigation of such objects seem to possess the promise to unravel further long-standing mysteries about the structure of the representation theory of space-time supersymmetry.

The adynkra in (7.1) can be tensored with the $\boxed{16}$ representation. This yields the following decomposition of $\mathcal{V}_\alpha$ at each level:

- Level-0: $\boxed{16}$

- Level-1: $\square_{\text{IR}} \;\oplus\; \yng(2)_{\text{IR}} \;\oplus\; \yng(3)_{\text{IR},-}$

- Level-2: $\boxed{16} \;\oplus\; \square\boxed{16}_{\text{IR}} \;\oplus\; \yng(2,1)^{\boxed{16}}_{\text{IR}} \;\oplus\; \yng(2,1)^{\boxed{16}}_{\text{IR}}$

- Level-3: $\yng(2)_{\text{IR}} \;\oplus\; \yng(3)_{\text{IR}} \;\oplus\; \yng(3,1)_{\text{IR}} \;\oplus\; \yng(2,1,1)_{\text{IR},+} \;\oplus\; \yng(3,1,1)_{\text{IR}}$

- Level-4: $\square\boxed{16}_{\text{IR}} \;\oplus\; \yng(2,1)^{\boxed{16}}_{\text{IR}} \;\oplus\; \yng(1,1,1,1)^{\boxed{16}}_{\text{IR},+} \;\oplus\; \yng(2,1)^{\boxed{16}}_{\text{IR}} \;\oplus\; (2)\,\yng(2,1)^{\boxed{16}}_{\text{IR}}$
  $\oplus\; \yng(2,1)^{\boxed{16}}_{\text{IR}} \;\oplus\; \yng(2,1,1)^{\boxed{16}}_{\text{IR}}$

- Level-5: $\yng(2)_{\text{IR},+} \;\oplus\; \yng(2,1)_{\text{IR}} \;\oplus\; \yng(2,1,1)_{\text{IR}} \;\oplus\; \yng(2,2)_{\text{IR}} \;\oplus\; (2)\,\yng(2,2,1)_{\text{IR},+} \;\oplus\; \yng(3,2)_{\text{IR}}$
  $\oplus\; \yng(3,2)_{\text{IR}} \;\oplus\; \yng(2,1,1,1)_{\text{IR},+} \;\oplus\; \yng(2,2,2)_{\text{IR},+} \;\oplus\; \yng(3,2,1)_{\text{IR}}$

- Level-6: $\oplus$ $\oplus$ $\oplus$ $\oplus$

$\oplus$ $\oplus$ $\oplus$ (2) $\oplus$ $\oplus$

$\oplus$

- Level-7: $\oplus$ $\oplus$ $\oplus$ $\oplus$ $\oplus$

$\oplus$ $\oplus$ (2) $\oplus$ $\oplus$ $\oplus$

$\oplus$ $\oplus$

- Level-8: $\oplus$ $\oplus$ (2) $\oplus$ $\oplus$

$\oplus$ (2) $\oplus$ $\oplus$ $\oplus$ $\oplus$

$\oplus$ $\oplus$

- Level-9: $\oplus$ $\oplus$ $\oplus$ $\oplus$ (2)

$\oplus$ $\oplus$ $\oplus$ $\oplus$ $\oplus$

$\oplus$ $\oplus$ $\oplus$

- Level-10:

- Level-11:

- Level-12:

- Level-13:

- Level-14:

- Level-15: $\cdot \oplus$

- Level-16: $\boxed{16}$

We can also apply $\mathcal{D}$ to the adynkrafield in (8.3),

$$\mathcal{D}\widehat{\mathcal{G}}(x) \;=\; \boxed{16}\,\Psi_\alpha(x) \;+\; (\ell)\left[\,\boxed{}_{\mathrm{IR}}\,\Phi_{\{\underline{a}_1\underline{b}_1\underline{c}_1\}}(x) \;+\; i\,\square_{\mathrm{IR}}\partial_{\underline{a}_1}\Phi(x)\right]$$

$$+ \ \frac{1}{2!}\,(\ell)^2 \left[\ \boxed{\begin{smallmatrix}\fbox{16}\\ \end{smallmatrix}}_{\text{IR}} \Psi_{\{\underline{a}_1\underline{b}_1\}}{}^\alpha(x) \ + \ i\,2\,\big(\fbox{16} \ + \ \Box\fbox{16}_{\text{IR}} \big)\,\partial_{\underline{a}_1}\Psi_\alpha(x) \right]$$

$$+ \ \frac{1}{3!}\,(\ell)^3 \left[\ \boxplus_{\text{IR}}\Phi_{\{\underline{a}_1\underline{b}_1,\underline{a}_2\underline{b}_2\}}(x) \ + \ \begin{smallmatrix}\Box\\\Box\\\Box\end{smallmatrix}_{\text{IR},-}\Phi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^+}(x) \right.$$

$$\left. + \ i\,3 \left(\begin{smallmatrix}\Box\\\Box\end{smallmatrix}_{\text{IR}} \ + \ \begin{smallmatrix}\Box\\\Box\\\Box\end{smallmatrix}_{\text{IR}} \ + \ \begin{smallmatrix}\boxplus\\\Box\end{smallmatrix}_{\text{IR}} \right)\,\partial_{\underline{a}_2}\Phi_{\{\underline{a}_1\underline{b}_1\underline{c}_1\}}(x) \right]$$

$$+ \ \frac{1}{4!}\,(\ell)^4 \left[\ \begin{smallmatrix}\fbox{16}\\\Box\\\Box\\\Box\end{smallmatrix}_{\text{IR},-}\Psi_{\{\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^+}{}^\alpha(x) \ + \ \boxed{\Box\fbox{16}}_{\text{IR}}\Psi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\}}{}^\alpha(x) \right.$$

$$\left. + \ i\,4 \left(\Box\fbox{16}_{\text{IR}} \ + \ \begin{smallmatrix}\fbox{16}\\\Box\end{smallmatrix}_{\text{IR}} \ + \ \begin{smallmatrix}\fbox{16}\\\Box\\\Box\end{smallmatrix}_{\text{IR}} \ + \ \begin{smallmatrix}\Box\fbox{16}\\\Box\end{smallmatrix}_{\text{IR}} \right)\,\partial_{\underline{a}_2}\Psi_{\{\underline{a}_1\underline{b}_1\}}{}^\alpha(x) \right]$$

$$+ \ \frac{1}{5!}\,(\ell)^5 \left[\ \begin{smallmatrix}\boxplus\\\boxplus\\\Box\end{smallmatrix}_{\text{IR},-}\Phi_{\{\underline{a}_2\underline{b}_2|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^+}(x) \ + \ \begin{smallmatrix}\boxplus\\\Box\end{smallmatrix}_{\text{IR}}\Phi_{\{\underline{a}_2,\underline{a}_3|\underline{a}_1\underline{b}_1\underline{c}_1\}}(x) \right.$$

$$+ \ i\,5 \left(\begin{smallmatrix}\boxplus\\\Box\end{smallmatrix}_{\text{IR}} \ + \ \begin{smallmatrix}\Box\\\boxplus\end{smallmatrix}_{\text{IR}} \ + \ \boxplus\boxplus_{\text{IR}} \right)\,\partial_{\underline{a}_3}\Phi_{\{\underline{a}_1\underline{b}_1,\underline{a}_2\underline{b}_2\}}(x)$$

$$\left. + \ i\,5 \left(\begin{smallmatrix}\Box\\\Box\\\Box\\\Box\end{smallmatrix}_{\text{IR},-} \ + \ \begin{smallmatrix}\boxplus\\\Box\\\Box\end{smallmatrix}_{\text{IR}} \ + \ \begin{smallmatrix}\boxplus\\\boxplus\\\Box\end{smallmatrix}_{\text{IR},-} \ + \ \begin{smallmatrix}\boxplus\boxplus\\\Box\end{smallmatrix}_{\text{IR},-} \right)\,\partial_{\underline{a}_3}\Phi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^+}(x) \right]$$

$$+ \ \frac{1}{6!}\,(\ell)^6 \left[\ \Box\Box\Box\fbox{16}_{\text{IR}}\Psi_{\{\underline{a}_1,\underline{a}_2,\underline{a}_3\}\,\alpha}(x) \ + \ \begin{smallmatrix}\Box\fbox{16}\\\boxplus\end{smallmatrix}_{\text{IR}}\Psi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\underline{c}_1\}}{}^\alpha(x) \right.$$

$$+ \ i\,6 \left(\begin{smallmatrix}\Box\fbox{16}\\\Box\\\Box\end{smallmatrix}_{\text{IR}} \ + \ \begin{smallmatrix}\Box\fbox{16}\\\Box\\\Box\end{smallmatrix}_{\text{IR},-} \right)\,\partial_{\underline{a}_2}\Psi_{\{\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^+}{}^\alpha(x)$$

$$+ \ i\,6 \left(\begin{smallmatrix}\Box\fbox{16}\\\Box\end{smallmatrix}_{\text{IR}} \ + \ \Box\Box\fbox{16}_{\text{IR}} \ + \ \begin{smallmatrix}\boxplus\fbox{16}\\ \end{smallmatrix}_{\text{IR}} \ + \ \begin{smallmatrix}\boxplus\Box\fbox{16}\\ \end{smallmatrix}_{\text{IR}} \right.$$

$$\left. + \ \begin{smallmatrix}\boxplus\fbox{16}\\\Box\end{smallmatrix}_{\text{IR}} \ + \ \boxplus\Box\fbox{16}_{\text{IR}} \right)\,\partial_{\underline{a}_3}\Psi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\}}{}^\alpha(x) \right]$$

$$+ \ \frac{1}{7!}\,(\ell)^7 \left[\ \Box\Box\Box\Box_{\text{IR}}\Phi_{\{\underline{a}_1,\underline{a}_2,\underline{a}_3,\underline{a}_4\}}(x) \ + \ \begin{smallmatrix}\boxplus\\\boxplus\end{smallmatrix}_{\text{IR}}\Phi_{\{\underline{a}_1\underline{b}_1\underline{c}_1,\underline{a}_2\underline{b}_2\underline{c}_2\}}(x) \right.$$

$$+ \ \square\!\!\square \ \Phi_{\{\underline{a}_2,\underline{a}_3|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\}}(x) \ + \ i\,7\Bigg(\ \square_{\mathrm{IR},-} \ + \ \square_{\mathrm{IR}} \ + \ \square_{\mathrm{IR},-}$$

$$+ \ \square_{\mathrm{IR},-} \ \Bigg)\,\partial_{\underline{a}_3}\,\Phi_{\{\underline{a}_2\underline{b}_2|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^+}(x) \ + \ i\,7\Bigg(\ \square_{\mathrm{IR}} \ + \ \square_{\mathrm{IR}}$$

$$+ \ \square_{\mathrm{IR}} \ + \ \square_{\mathrm{IR}} \ + \ \square_{\mathrm{IR}} \ \Bigg)\,\partial_{\underline{a}_3}\,\Phi_{\{\underline{a}_2,\underline{a}_3|\underline{a}_1\underline{b}_1\underline{c}_1\}}(x)\ \Bigg]$$

$$+ \ \frac{1}{8!}\,(\ell)^8\,\Bigg[\ \boxed{\overline{16}}_{\mathrm{IR}}\,\Psi_{\{\underline{a}_1,\underline{a}_2,\underline{a}_3\}}{}^{\alpha}(x) \ + \ \boxed{16}_{\mathrm{IR}}\,\Psi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\underline{c}_1\}\alpha}(x)$$

$$+ \ i\,8\Bigg(\ \boxed{16}_{\mathrm{IR}} \ + \ \boxed{16}_{\mathrm{IR}} \ + \ \boxed{16}_{\mathrm{IR}}$$

$$+ \ \boxed{16}_{\mathrm{IR}} \ \Bigg)\,\partial_{\underline{a}_4}\,\Psi_{\{\underline{a}_1,\underline{a}_2,\underline{a}_3\}\alpha}(x) \ + \ i\,8\Bigg(\ \boxed{16}_{\mathrm{IR}} \ + \ \boxed{16}_{\mathrm{IR}}$$

$$+ \ \boxed{16}_{\mathrm{IR}} \ + \ \boxed{16}_{\mathrm{IR}} \ + \ \boxed{16}_{\mathrm{IR}} \ + \ \boxed{16}_{\mathrm{IR}} \ \Bigg)\,\partial_{\underline{a}_3}\,\Psi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\underline{c}_1\}}{}^{\alpha}(x)\ \Bigg]$$

$$+ \ \frac{1}{9!}\,(\ell)^9\,\Bigg[\ \square_{\mathrm{IR},+}\,\Phi_{\{\underline{a}_2\underline{b}_2|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^-}(x) \ + \ \square_{\mathrm{IR}}\,\widehat{\Phi}_{\{\underline{a}_2,\underline{a}_3|\underline{a}_1\underline{b}_1\underline{c}_1\}}(x)$$

$$+ \ i\,9\Bigg(\ \square_{\mathrm{IR}} \ + \ \square_{\mathrm{IR}} \ + \ \square_{\mathrm{IR}} \ \Bigg)\,\partial_{\underline{a}_5}\,\Phi_{\{\underline{a}_1,\underline{a}_2,\underline{a}_3,\underline{a}_4\}}(x)$$

$$+ \ i\,9\Bigg(\ \square_{\mathrm{IR}} \ + \ \square_{\mathrm{IR}} \ + \ \square_{\mathrm{IR}} \ \Bigg)\,\partial_{\underline{a}_3}\,\Phi_{\{\underline{a}_1\underline{b}_1\underline{c}_1,\underline{a}_2\underline{b}_2\underline{c}_2\}}(x)$$

$$+ \ i\,9\Bigg(\ \square_{\mathrm{IR}} \ + \ \square_{\mathrm{IR}} \ + \ \square_{\mathrm{IR},-} \ + \ \square_{\mathrm{IR},+} \ + \ \square_{\mathrm{IR}}$$

$$+ \ \square_{\mathrm{IR}} \ \Bigg)\,\partial_{\underline{a}_4}\,\Phi_{\{\underline{a}_2,\underline{a}_3|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\}}(x)\ \Bigg]$$

$$+ \ \frac{1}{10!}\,(\ell)^{10}\,\Bigg[\ \boxed{16}_{\mathrm{IR},+}\,\Psi_{\{\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^-\alpha}(x) \ + \ \boxed{16}_{\mathrm{IR}}\,\Psi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\}\alpha}(x)$$

$$+\, i\,10\left(\;\boxed{\;\;16\;}_{\text{IR}}\;+\;\boxed{\;\;\;16\;}_{\text{IR}}\;+\;\boxed{\;\;\;\;16\;}_{\text{IR}}\right.$$

$$\left.+\;\boxed{\;\;16\;}_{\text{IR}}\right)\partial_{\underline{a}_4}\Psi_{\{\underline{a}_1,\underline{a}_2,\underline{a}_3\}}{}^{\alpha}(x)\;+\;i\,10\left(\;\boxed{16}_{\text{IR}}\;+\;\boxed{16}_{\text{IR}}\;+\;\boxed{16}_{\text{IR}}\right.$$

$$\left.+\;\boxed{16}_{\text{IR}}\;+\;\boxed{16}_{\text{IR}}\;+\;\boxed{16}_{\text{IR}}\right)\partial_{\underline{a}_3}\Psi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\underline{c}_1\}\alpha}(x)\Bigg]$$

$$+\;\frac{1}{11!}\,(\ell)^{11}\left[\;\boxed{\;\;}_{\text{IR}}\,\widehat{\Phi}_{\{\underline{a}_1\underline{b}_1,\underline{a}_2\underline{b}_2\}}(x)\;+\;\boxed{\;\;}_{\text{IR},+}\;\Phi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^-}(x)\right.$$

$$+\;i\,11\left(\;\boxed{\;\;}_{\text{IR},+}\;+\;\boxed{\;\;}_{\text{IR}}\;+\;\boxed{\;\;}_{\text{IR},+}\;+\;\boxed{\;\;}_{\text{IR},+}\;\right)$$

$$\times\,\partial_{\underline{a}_3}\Phi_{\{\underline{a}_2\underline{b}_2|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^-}(x)$$

$$+\;i\,11\left(\;\boxed{\;\;}_{\text{IR}}\;+\;\boxed{\;\;}_{\text{IR}}\;+\;\boxed{\;\;}_{\text{IR}}\;+\;\boxed{\;\;}_{\text{IR}}\right.$$

$$\left.+\;\boxed{\;\;}_{\text{IR}}\;\right)\partial_{\underline{a}_4}\widehat{\Phi}_{\{\underline{a}_2,\underline{a}_3|\underline{a}_1\underline{b}_1\underline{c}_1\}}(x)\Bigg]$$

$$+\;\frac{1}{12!}\,(\ell)^{12}\left[\;\boxed{16}_{\text{IR}}\,\Psi_{\{\underline{a}_1\underline{b}_1\}\alpha}(x)\right.$$

$$+\;i\,12\left(\;\boxed{16}_{\text{IR}}\;+\;\boxed{16}_{\text{IR},+}\;\right)\partial_{\underline{a}_2}\Psi_{\{\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^-\alpha}(x)$$

$$+\;i\,12\left(\;\boxed{16}_{\text{IR}}\;+\;\boxed{\;\;16}_{\text{IR}}\;+\;\boxed{16}_{\text{IR}}\;+\;\boxed{16}_{\text{IR}}\;+\;\boxed{16}_{\text{IR}}\right.$$

$$\left.+\;\boxed{\;\;16}_{\text{IR}}\;\right)\partial_{\underline{a}_3}\Psi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\}\alpha}(x)\Bigg]$$

$$+\;\frac{1}{13!}\,(\ell)^{13}\left[\;\boxed{\;\;}_{\text{IR}}\,\widehat{\Phi}_{\{\underline{a}_1\underline{b}_1\underline{c}_1\}}(x)\right.$$

$$+\;i\,13\left(\;\boxed{\;\;}_{\text{IR}}\;+\;\boxed{\;\;}_{\text{IR}}\;+\;\boxed{\;\;}_{\text{IR}}\;\right)\partial_{\underline{a}_3}\widehat{\Phi}_{\{\underline{a}_1\underline{b}_1,\underline{a}_2\underline{b}_2\}}(x)$$

$$
+ \; i\,13 \left(\young_{\mathrm{IR},+} + \young_{\mathrm{IR}} + \young_{\mathrm{IR},+} + \young_{\mathrm{IR},+} \right)
$$

$$
\times \; \partial_{\underline{a}_3} \Phi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^-}(x) \Bigg]
$$

$$
+ \; \tfrac{1}{14!}\,(\ell)^{14}\left[\,\boxed{\overline{16}}\,\Psi^\alpha(x)\right.
$$

$$
+ \; i\,14\left(\boxed{}\,\boxed{16}_{\mathrm{IR}} + \young\,\boxed{16}_{\mathrm{IR}} + \young\,^{\boxed{16}}_{\mathrm{IR}} + \young\,\boxed{16}_{\mathrm{IR}}\right)\partial_{\underline{a}_2}\Psi_{\{\underline{a}_1\underline{b}_1\}\alpha}(x)\Bigg]
$$

$$
+ \; \tfrac{1}{15!}\,(\ell)^{15}\left[\,\widehat{\Phi}(x) \; + \; i\,15\left(\boxed{}_{\mathrm{IR}} + \young_{\mathrm{IR}} + \young_{\mathrm{IR}}\right)\partial_{\underline{a}_2}\widehat{\Phi}_{\{\underline{a}_1\underline{b}_1\underline{c}_1\}}(x)\right]
$$

$$
+ \; i\,\tfrac{1}{15!}\,(\ell)^{16}\left(\boxed{16} + \boxed{}\,\boxed{16}_{\mathrm{IR}}\right)\partial_{\underline{a}_1}\Psi^\alpha(x) \; .
$$

(8.5)

It can be seen the effect of the operator $\mathcal{D}$ upon calculating $\mathcal{D}\,\widehat{\mathcal{G}}(x)$ is similar to the result that is found for the tensoring calculation $\boxed{16}\;\widehat{\mathcal{G}}(x)$. However, the two calculations yield different results. One way to quantify the difference is to define

$$
(8.6) \qquad \Delta_{WZg}(n) \;=\; \frac{17! \times n}{(n+1)!\,(16-n)!} \qquad\qquad ,
$$

so that at each value of n the quantity $\Delta_{WZg}(n)$ counts the number of degrees of freedom in the difference $[\;\boxed{16}\; \text{-}\; \mathcal{D}\;]\;\widehat{\mathcal{G}}(x)$ at Level-n[13]. A further calculation reveals

$$
(8.7) \qquad \sum_{n=1}^{16}\Delta_{WZg}(n) \;=\; 983,041 \;=\; [(16-1)\times(65,536)] \; + \; 1 \; .
$$

If we regard $\boxed{16}\;\widehat{\mathcal{G}}(x)$ as a connection adynkrafield and $\mathcal{D}\widehat{\mathcal{G}}(x)$ as its gauge transformation, then 491,521 is the number bosonic components which is

---

[13] The way to understand the reason for the form of (8.7) is to recall the lowest component field of $\mathcal{G}$ must describe the component level gauge parameter that is present even in the WZ gauge and hence the mismatch between the naive equality of bosons versus fermions.

not equal to the 491,520 fermionic components contained in the adynkrafield connection in a Wess-Zumino gauge with respect this gauge transformation.

8.1. From symbolic notation to tensor notation

As we have mentioned, the fields that are shown in (8.3) have the indices on the field contracted with the "invisible" indices on the YT's. For this equation, the contractions should be relatively straight forward to surmise. Instead with (8.5) these contractions may not appear so obvious. Thus, in this subsection, time will be spent showing some example in the hope the general case will be made clear.

Let us begin with the second order terms on the second line of (8.5)

(8.8)
$$\left[\mathcal{D}\widehat{\mathcal{G}}(x) \right]_{(\ell)^2} = + \frac{1}{2!}(\ell)^2 \left[\boxed{}^{\boxed{16}}_{\text{IR}} \Psi_{\{\underline{a}_1\underline{b}_1\}}{}^\alpha + i\,2\,(\boxed{16} + \boxed{16}_{\text{IR}})\,\partial_{\underline{a}_1}\Psi_\alpha \right] .$$

On the leading term involving $\Psi_{\{\underline{a}_1\underline{b}_1\}}{}^\alpha(x)$, the $\underline{a}_1$-index, and the $\underline{b}_1$-index are contracted with the two invisible bosonic indices of $\boxminus$ and the α-index is contracted with the invisible fermionic index of $\boxed{16}$. For the remaining terms there are subtleties to consider. Our deliberations begin by expanding the final two terms in (8.8)

(8.9) $\qquad (\,\boxed{16}\; + \;\boxed{16}_{\text{IR}}\,)\,\partial_{\underline{a}_1}\Psi_\alpha = \boxed{16}\,(\,\partial_{\underline{a}_1}\Psi_\alpha\,) + \boxed{16}_{\text{IR}}\,(\,\partial_{\underline{a}_1}\Psi_\alpha\,)$.

Next we introduce the relation

(8.10) $\qquad \partial_{\underline{a}_1}\Psi_\alpha = [\,\partial_{\underline{a}_1}\Psi_\alpha - k_1\,(\sigma_{\underline{a}_1})_{\alpha\delta}(\sigma^{\underline{b}_1})^{\delta\gamma}\,\partial_{\underline{b}_1}\Psi_\gamma\,]$
$$+ k_1\,(\sigma_{\underline{a}_1})_{\alpha\delta}(\sigma^{\underline{b}_1})^{\delta\gamma}\,\partial_{\underline{b}_1}\Psi_\gamma$$

which implies the equations in (8.11) and (8.12)

(8.11) $\qquad \boxed{16}\,\partial_{\underline{a}_1}\Psi_\alpha = \boxed{16}\,[\,\partial_{\underline{a}_1}\Psi_\alpha - k_1\,(\sigma_{\underline{a}_1})_{\alpha\delta}(\sigma^{\underline{b}_1})^{\delta\gamma}\,\partial_{\underline{b}_1}\Psi_\gamma\,]$
$$+ k_1\,\boxed{16}\,(\sigma_{\underline{a}_1})_{\alpha\delta}(\sigma^{\underline{b}_1})^{\delta\gamma}\,\partial_{\underline{b}_1}\Psi_\gamma \ ,$$

and

(8.12) $\qquad \boxed{16}_{\text{IR}}\,\partial_{\underline{a}_1}\Psi_\alpha = \boxed{16}_{\text{IR}}\,[\,\partial_{\underline{a}_1}\Psi_\alpha - k_1\,(\sigma_{\underline{a}_1})_{\alpha\delta}(\sigma^{\underline{b}_1})^{\delta\gamma}\,\partial_{\underline{b}_1}\Psi_\gamma\,]$
$$+ k_1\,\boxed{16}_{\text{IR}}\,(\sigma_{\underline{a}_1})_{\alpha\delta}(\sigma^{\underline{b}_1})^{\delta\gamma}\,\partial_{\underline{b}_1}\Psi_\gamma \ .$$

Our conventions for 10D, $\mathcal{N} = 1$ superspace have previously been given in [18] where it was presented that we use a "mostly plus" Minkowski metric

and define a 10D set of Pauli matrices by

$$(8.13) \qquad (\sigma^{\underline{a}})_{\alpha\beta}\,(\sigma^{\underline{b}})^{\beta\gamma} + (\sigma^{\underline{b}})_{\alpha\beta}\,(\sigma^{\underline{a}})^{\beta\gamma} = 2\,\eta^{\underline{ab}}\,\delta_\alpha{}^\gamma \quad,$$

whereby this equation implies the following result

$$(8.14) \qquad (\sigma^{\underline{a}})_{\alpha\beta}\,(\sigma^{\underline{b}})^{\beta\gamma} \;=\; \eta^{\underline{ab}}\,\delta_\alpha{}^\gamma \;+\; (\sigma^{\underline{ab}})_\alpha{}^\gamma \quad,$$

for our calculations. Contraction on the vector indices in (8.14) yields

$$(8.15) \qquad (\sigma^{\underline{a}})_{\alpha\beta}(\sigma_{\underline{a}})^{\beta\gamma} \;=\; 10\,\delta_\alpha{}^\gamma \quad.$$

Thus, if we define $[\mathcal{P}_1]_{\underline{a}}{}^{\underline{b}}{}_\alpha{}^\gamma$ and $[\mathcal{P}_2]_{\underline{a}}{}^{\underline{b}}{}_\alpha{}^\gamma$ via the equations

$$(8.16) \qquad \begin{aligned} [\mathcal{P}_1]_{\underline{a}}{}^{\underline{b}}{}_\alpha{}^\gamma &\;=\; \left[\delta_{\underline{a}}{}^{\underline{b}}\,\delta_\alpha{}^\gamma - \tfrac{1}{10}\,(\sigma_{\underline{a}})_{\alpha\delta}(\sigma^{\underline{b}})^{\delta\gamma}\right] \;, \\ [\mathcal{P}_2]_{\underline{a}}{}^{\underline{b}}{}_\alpha{}^\gamma &\;=\; \tfrac{1}{10}\,(\sigma_{\underline{a}})_{\alpha\delta}(\sigma^{\underline{b}})^{\delta\gamma} \end{aligned}$$

we obtain

$$(8.17) \qquad \begin{aligned} [\mathcal{P}_1]_{\underline{a}}{}^{\underline{c}}{}_\alpha{}^\delta\,[\mathcal{P}_1]_{\underline{c}}{}^{\underline{b}}{}_\delta{}^\gamma &\;=\; [\mathcal{P}_1]_{\underline{a}}{}^{\underline{b}}{}_\alpha{}^\gamma \;, \\ [\mathcal{P}_2]_{\underline{a}}{}^{\underline{c}}{}_\alpha{}^\delta\,[\mathcal{P}_2]_{\underline{c}}{}^{\underline{b}}{}_\delta{}^\gamma &\;=\; [\mathcal{P}_2]_{\underline{a}}{}^{\underline{b}}{}_\alpha{}^\gamma \;, \\ [\mathcal{P}_1]_{\underline{a}}{}^{\underline{c}}{}_\alpha{}^\delta\,[\mathcal{P}_2]_{\underline{c}}{}^{\underline{b}}{}_\delta{}^\gamma &\;=\; 0 \;. \end{aligned}$$

Upon choosing $k_1 = \tfrac{1}{10}$, we can rewrite the results shown in (8.11) and (8.12) in the forms

$$(8.18) \qquad \boxed{16}\,\partial_{\underline{a}_1}\Psi_\alpha \;=\; \boxed{16}\,[\mathcal{P}_1]_{\underline{a}_1}{}^{\underline{c}_1}{}_\alpha{}^\delta\,[\partial_{\underline{c}_1}\Psi_\delta] + \boxed{16}\,[\mathcal{P}_2]_{\underline{a}_1}{}^{\underline{c}_1}{}_\alpha{}^\delta\,[\partial_{\underline{c}_1}\Psi_\delta] \;,$$

and

$$(8.19) \qquad \boxed{\;}\boxed{16}_{\mathrm{IR}}\partial_{\underline{a}_1}\Psi_\alpha \;=\; \boxed{\;}\boxed{16}_{\mathrm{IR}}\,[\mathcal{P}_1]_{\underline{a}_1}{}^{\underline{c}_1}{}_\alpha{}^\delta\,[\partial_{\underline{c}_1}\Psi_\delta]$$
$$+\; \boxed{\;}\boxed{16}_{\mathrm{IR}}\,[\mathcal{P}_2]_{\underline{a}_1}{}^{\underline{c}_1}{}_\alpha{}^\delta\,[\partial_{\underline{c}_1}\Psi_\delta] \;.$$

The projection operators $[\mathcal{P}_1]_{\underline{a}}{}^{\underline{b}}{}_\alpha{}^\beta$ and $[\mathcal{P}_2]_{\underline{a}}{}^{\underline{b}}{}_\alpha{}^\beta$ are precisely the ones that separate the irreducible projections of $\partial_{\underline{b}}\Psi_\beta$ into its "sigma-traceless" part and its "pure sigma trace" part. Thus, it follows that (8.18) and (8.19) can

be rewritten in the simpler forms

$$(8.20) \qquad \boxed{16}\, \partial_{\underline{a}_1}\, \Psi_\alpha \;=\; \boxed{16}\, [\mathcal{P}_2]_{\underline{a}_1}{}^{\underline{b}_1}{}_\alpha{}^\beta\, [\partial_{\underline{b}_1}\, \Psi_\beta] \quad,$$

and

$$(8.21) \qquad \boxed{}\boxed{16}_{\mathrm{IR}}\partial_{\underline{a}_1}\, \Psi_\alpha \;=\; \boxed{}\boxed{16}_{\mathrm{IR}}\, [\mathcal{P}_1]_{\underline{a}_1}{}^{\underline{b}_1}{}_\alpha{}^\beta\, [\partial_{\underline{b}_1}\, \Psi_\beta] \quad.$$

This set of calculations yields a very general rule when contracting the indices on a YT with a following expression involving fermionic fields. Namely, when the number of blue boxes in the YT does not match the number of bosonic indices in the expression, this is due to the appearance of "sigma traces" in the fermionic expression.

We now turn to the third order terms on the second line of (8.5)

$$\left[\mathcal{D}\widehat{\mathcal{G}}(x)\right]_{(\ell)^3} \;=\; +\, \frac{1}{3!}\, (\ell)^3 \left[\;\boxminus_{\mathrm{IR}}\; \Phi_{\{\underline{a}_1\underline{b}_1,\underline{a}_2\underline{b}_2\}}(x) \right.$$

$$+\; \sqboxedcol_{\mathrm{IR},-} \; \Phi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^+}(x)$$

$$(8.22) \qquad +\; i\,3\left(\boxminus_{\mathrm{IR}} \;+\; \sqboxed_{\mathrm{IR}} \;+\; \boxminus\boxminus_{\mathrm{IR}} \right) \partial_{\underline{a}_2}\, \Phi_{\{\underline{a}_1\underline{b}_1\underline{c}_1\}}(x) \left. \vphantom{\frac11}\right].$$

On the leading term involving $\Phi_{\{\underline{a}_1\underline{b}_1,\underline{a}_2\underline{b}_2\}}(x)$, the $\underline{a}_1$-index, and the $\underline{b}_1$-index are contracted with the two invisible bosonic indices in the first column of the preceding YT, and the $\underline{a}_2$-index, and the $\underline{b}_2$-index are contracted with the two invisible bosonic indices in the second column of the preceding YT.

On the second term involving $\Phi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^+}(x)$, the $\underline{a}_1$, ..., $\underline{e}_1$ indices are contracted with the five invisible bosonic indices in the first column of the preceding YT, and the $\underline{a}_2$-index is contracted with the single box in the second column of the preceding YT.

The really interesting terms appear on the second line of (8.22). Here our deliberations begin by expanding the final three terms,

$$(8.23)$$

$$\left(\boxminus_{\mathrm{IR}} \;+\; \sqboxed_{\mathrm{IR}} \;+\; \boxminus\boxminus_{\mathrm{IR}} \right) \partial_{\underline{a}_2}\, \Phi_{\{\underline{a}_1\underline{b}_1\underline{c}_1\}}(x)$$

$$= \boxed{}_{\text{IR}} \, \partial_{\underline{a}_2} \, \Phi_{\{\underline{a}_1 \underline{b}_1 \underline{c}_1\}}(x) \; + \; \boxed{}_{\text{IR}} \, \partial_{\underline{a}_2} \, \Phi_{\{\underline{a}_1 \underline{b}_1 \underline{c}_1\}}(x) \; + \; \boxed{}_{\text{IR}} \, \partial_{\underline{a}_2} \, \Phi_{\{\underline{a}_1 \underline{b}_1 \underline{c}_1\}}(x) \quad .$$

Here the only subtlety involves the first term.

On the leading term involving $\partial_{\underline{a}_2} \, \Phi_{\{\underline{a}_1 \underline{b}_1 \underline{c}_1\}}(x)$, it is immediately clear there are only two invisible bosonic indices on the YT, but there are four bosonic indices on the spatial derivative acting on the field that follows the tableau. The matching of indices can only be reconciled if two of the indices are "contracted away." This means we must have

$$(8.24) \qquad \boxed{}_{\text{IR}} \, \partial_{\underline{a}_2} \, \Phi_{\{\underline{a}_1 \underline{b}_1 \underline{c}_1\}}(x) \; = \; \boxed{}_{\text{IR}} \, \eta^{\underline{a}_1 \underline{a}_2} \, \partial_{\underline{a}_2} \, \Phi_{\{\underline{a}_1 \underline{b}_1 \underline{c}_1\}}(x) \quad .$$

with the $\underline{b}_1$ and $\underline{c}_1$ indices contracted with the preceding YT. It should be noted that the antisymmetry of the the indices of the field $\Phi_{\{\underline{a}_1 \underline{b}_1 \underline{c}_1\}}(x)$ implies that the choice of which index is contracted with the partial derivative is immaterial.

For the second term involving $\partial_{\underline{a}_2} \, \Phi_{\{\underline{a}_1 \underline{b}_1 \underline{c}_1\}}(x)$, the process is straightforward. Each bosonic index in the expression $\partial_{\underline{a}_2} \, \Phi_{\{\underline{a}_1 \underline{b}_1 \underline{c}_1\}}(x)$ is contracted with an invisible index of one of the boxes in the YT.

Finally, for the third and last term involving $\partial_{\underline{a}_2} \, \Phi_{\{\underline{a}_1 \underline{b}_1 \underline{c}_1\}}(x)$, the index on the partial derivative is contracted with the invisible index of the box in the second column. The remaining indices $\underline{a}_1$, $\underline{b}_1$, and $\underline{c}_1$ are each contracted with one of the invisible indices associated with the boxes in the first column of the preceding YT.

This set of calculations yields a very general rule, analogous to the one appearing below (8.21), when contracting the indices on a YT with a following expression involving bosonic fields. Namely, when the number of blue boxes in the YT does not match the number of bosonic indices in the expression, this is due to the appearance of the Minkowski metric in the bosonic expression.

8.2. From adynkrafields back to 1D adinkras

A final amusing matter is that having reached the introduction of adynkrafields, we can take a limit where the higher dimensional adynkrafields are forced into the form of 1D, $N = 16$ valise adinkras! The is done by imposing the condition that all of the field variables depend solely on a time-like coordinate τ and the imposition of the condition that $(\ell)^2 =$

1. This leads to

$$
\begin{aligned}
\widehat{\mathcal{G}}_{Adnk}(\tau) = \Bigg\{ & \Phi(\tau) \;+\; \tfrac{1}{2}\,\yng(1,1)_{\text{IR}}\, \Phi_{\{\underline{a}_1\underline{b}_1\underline{c}_1\}}(\tau) \;+\; \tfrac{1}{4!}\,\yng(2,2)_{\text{IR}}\, \Phi_{\{\underline{a}_1\underline{b}_1,\underline{a}_2\underline{b}_2\}}(\tau) \\[2mm]
& +\; \tfrac{1}{4!}\,\yng(2,1,1,1)_{\text{IR},-}\, \Phi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^{+}}(\tau) \\[2mm]
& +\; \tfrac{1}{6!}\,\yng(2,2,1,1)_{\text{IR},-}\, \Phi_{\{\underline{a}_2\underline{b}_2|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^{+}}(\tau) \;+\; \tfrac{1}{6!}\,\yng(3,2)_{\text{IR}}\, \Phi_{\{\underline{a}_2,\underline{a}_3|\underline{a}_1\underline{b}_1\underline{c}_1\}}(\tau) \\[2mm]
& +\; \tfrac{1}{8!}\,\yng(4)_{\text{IR}}\, \Phi_{\{\underline{a}_1,\underline{a}_2,\underline{a}_3,\underline{a}_4\}}(\tau) \;+\; \tfrac{1}{8!}\,\yng(2,2,2)_{\text{IR}}\, \Phi_{\{\underline{a}_1\underline{b}_1\underline{c}_1,\underline{a}_2\underline{b}_2\underline{c}_2\}}(\tau) \\[2mm]
& +\; \tfrac{1}{8!}\,\yng(3,2,1)_{\text{IR}}\, \Phi_{\{\underline{a}_2,\underline{a}_3|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\}}(\tau) \;+\; \tfrac{1}{10!}\,\yng(3,3,1)_{\text{IR}}\, \widehat{\Phi}_{\{\underline{a}_2,\underline{a}_3|\underline{a}_1\underline{b}_1\underline{c}_1\}}(\tau) \\[2mm]
& +\; \tfrac{1}{10!}\,\yng(2,2,2,1)_{\text{IR},+}\, \Phi_{\{\underline{a}_2\underline{b}_2|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^{-}}(\tau) \;+\; \tfrac{1}{12!}\,\yng(2,2)_{\text{IR}}\, \widehat{\Phi}_{\{\underline{a}_1\underline{b}_1,\underline{a}_2\underline{b}_2\}}(\tau) \\[2mm]
& +\; \tfrac{1}{12!}\,\yng(2,1,1,1)_{\text{IR},+}\, \Phi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^{-}}(\tau) \;+\; \tfrac{1}{14!}\,\yng(1,1)_{\text{IR}}\, \widehat{\Phi}_{\{\underline{a}_1\underline{b}_1\underline{c}_1\}}(\tau) \\[2mm]
& +\; \tfrac{1}{16!}\,\widehat{\Phi}(\tau) \Bigg\} \;+\; \ell\,\Bigg\{ \boxed{16}\,\Psi_\alpha(\tau) \;+\; \tfrac{1}{3!}\,\yngsixteen(2,1)_{\text{IR}}\, \Psi_{\{\underline{a}_1\underline{b}_1\}}{}^{\alpha}(\tau) \\[2mm]
& +\; \tfrac{1}{5!}\,\yngsixteen(1,1,1,1)_{\text{IR},-}\, \Psi_{\{\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^{+}}{}^{\alpha}(\tau) \;+\; \tfrac{1}{5!}\,\yngsixteen(2,2)_{\text{IR}}\, \Psi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\}}{}^{\alpha}(\tau) \\[2mm]
& +\; \tfrac{1}{7!}\,\yngsixteen(3)_{\text{IR}}\, \Psi_{\{\underline{a}_1,\underline{a}_2,\underline{a}_3\}\alpha}(\tau) \;+\; \tfrac{1}{7!}\,\yngsixteen(2,2,1)_{\text{IR}}\, \Psi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\underline{c}_1\}}{}^{\alpha}(\tau) \\[2mm]
& +\; \tfrac{1}{9!}\,\yngsixteen(3)_{\text{IR}}\, \Psi_{\{\underline{a}_1,\underline{a}_2,\underline{a}_3\}}{}^{\alpha}(\tau) \;+\; \tfrac{1}{9!}\,\yngsixteen(2,1,1)_{\text{IR}}\, \Psi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\underline{c}_1\}\alpha}(\tau) \\[2mm]
& +\; \tfrac{1}{11!}\,\yngsixteen(1,1,1,1)_{\text{IR},+}\, \Psi_{\{\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^{-}\alpha}(\tau) \;+\; \tfrac{1}{11!}\,\yngsixteen(2,1)_{\text{IR}}\, \Psi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\}\alpha}(\tau)
\end{aligned}
$$

$$(8.25) \qquad + \ \frac{1}{13!} \ \boxed{\begin{array}{c} \boxed{16} \\ \end{array}}_{\text{IR}} \ \Psi_{\{\underline{a}_1 \underline{b}_1\}\alpha}(\tau) \ + \ \frac{1}{15!} \ \boxed{16} \ \Psi^{\alpha}(\tau) \Bigg\} \ \ ,$$

which can be simplified further to eliminate all the factors involving the factorial function by rescaling the field variables appropriately.

This 1D, N = 16 valise adinkra system clearly contains 32,768 bosons and 32,768 fermions. It also contains the information associated with the Lorentz representations (via the YT's) of the original 10D, $\mathcal{N} = 1$ scalar supermultiplet for which it is the hologram. Application of the $\mathcal{D}$ operator to this expansion shown above while retaining terms only up to order (ℓ) will permit the derivations of the $\mathcal{GR}$ (d, N) matrices [40, 41] associated with this system.

In the work of [42], the inaugural discussion relating supermultiplets in greater than 1D to those in 1D was given. There was a portion of the derivation that was not explicitly presented. Let us recall from this past work, valise adinkra systems were described in the following manner.

A set of bosonic fields Φ_i and fermionic fields $\Psi_{\hat{k}}$ (where the index i takes on values for one to any integer d and the index $\hat{k}$ ranges of the same values) describe valise adinkra systems. Furthermore, two sets of matrices $(\mathrm{L_I})_{i\,\hat{k}}$ and $(\mathrm{R_I})_{\hat{k}\,i}$ are also introduced where the index I ranges over the integers, but its maximum value N is not necessarily restricted to be the same as that of the indices i and $\hat{k}$. The following conditions may be imposed on the matrices

$$(8.26) \qquad \begin{aligned} (\mathrm{L_I})_i{}^{\hat{j}} \, (\mathrm{R_J})_{\hat{j}}{}^{k} + (\mathrm{L_J})_i{}^{\hat{j}} \, (\mathrm{R_I})_{\hat{j}}{}^{k} &= 2\,\delta_{\mathrm{IJ}}\,\delta_i{}^{k} \ \ , \\ (\mathrm{R_J})_{\hat{i}}{}^{j} \, (\mathrm{L_I})_j{}^{\hat{k}} + (\mathrm{R_I})_{\hat{i}}{}^{j} \, (\mathrm{L_J})_j{}^{\hat{k}} &= 2\,\delta_{\mathrm{IJ}}\,\delta_{\hat{i}}{}^{\hat{k}} \ \ , \end{aligned}$$

$$(8.27) \qquad\qquad (\mathrm{R_I})_{\hat{j}}{}^{k}\,\delta_{ik} = (\mathrm{L_I})_i{}^{\hat{k}}\,\delta_{\hat{j}\hat{k}} \ \ ,$$

along with the following differential equations being imposed on Φ_i and $\Psi_{\hat{k}}$

$$(8.28) \qquad \mathrm{D_I}\Phi_i \ = \ i\,(\mathrm{L_I})_{i\,\hat{k}}\,\Psi_{\hat{k}} \ \ , \quad \mathrm{D_I}\Psi_{\hat{k}} \ = \ (\mathrm{R_I})_{\hat{k}\,i}\,\frac{d}{d\tau}\,\Phi_i \ \ .$$

so the equations (8.26), (8.27), and (8.28) uniformly imply the operator equation is satisfied

$$(8.29) \qquad D_I\, D_J \; + \; D_J\, D_I \; = \; i\, 2\, \delta_{IJ} \frac{d}{d\tau} \quad ,$$

on Φ_i and $\Psi_{\hat{k}}$. The fields Φ_i and $\Psi_{\hat{k}}$ can be related to the 1D fields in (8.25) via

$$(8.30) \qquad \begin{aligned} \Phi_i(\tau) \;&=\; \big\{\, \Phi(\tau),\, \Phi_{\{\underline{a_1}\underline{b_1}\underline{c_1}\}}(\tau),\, \Phi_{\{\underline{a_1}\underline{b_1},\underline{a_2}\underline{b_2}\}}(\tau),\, \dots \,\big\} \quad , \\ \Psi_{\hat{k}}(\tau) \;&=\; \big\{\, \Psi_\alpha(\tau),\, \Psi_{\{\underline{a_1}\underline{b_1}\}}{}^{\alpha}(\tau),\, \Psi_{\{\underline{a_1}\underline{b_1}\underline{c_1}\underline{d_1}\underline{e_1}\}^+}{}^{\alpha}(\tau),\, \dots \,\big\} \quad . \end{aligned}$$

Currently, it is not clear how to construct the D_I operators from $\mathcal{D}$. This will be the subject of a future investigation.

The algebra for the L-matrices and R-matrices in (8.26) defines the $\mathcal{GR}(d,\, N)$ algebra or the "Garden Algebra" (d, N). In the present context d = 32,768 and $N = 16$. In future investigations (as it will be possible to study the case where the ranges of i and $\hat{k}$ covers 1, ..., 32768, and the range of I covers 1, ..., 16), derivation will uncover how the (L_I) and (R_I) matrices holographically store the information of the YT's concerning the Lorentz representations of the fields in (8.25). This is the portion not undertaken in [42].

8.3. Adynkras, and links between nodes

It is clear from the presentation in Equation (7.1) for $\mathcal{V}$ (as well as the one on pages 57 - 59 for $\boxed{\color{red}16}\,\mathcal{V}$) that each adinkra associates some Level numbers with sets of YT's. Furthermore, the YT's are partitioned into two classes. The BYT's possess no red boxes while the SYT's possess one red box. Let us introduce notational devices for this division. We will use the symbol $\{\mathcal{R}^{(i)}\}_p$ to denote the "i-th" BYT at Level-p. It should be noted that the range of the index "i" depends on the value of p. In a similar manner, we will use the symbol $\{\mathcal{R}^{(i)}\}_p$ to denote the "i-th" SYT at Level-p. Once more it should be noted that the range of the index "i-th" depends on the value of p.

Next, we introduce four coefficients $c^{+(i)}_{\{\mathcal{R}^{(j)}\}_{p+1}}$, $c^{-(i)}_{\{\mathcal{R}^{(j)}\}_{p-1}}$, $c^{+(j)}_{\{\mathcal{R}^{(i)}\}_{p+1}}$, and $c^{-(j)}_{\{\mathcal{R}^{(i)}\}_{p-1}}$ which are determined by examining properties of the adynkra. In particular, there are four calculations to be implemented, and these are respectively

$$(8.31) \qquad c^{+(i)}_{\{\mathcal{R}^{(j)}\}_{p+1}} \; = \; \mathcal{F}_1 \left[\left(\boxed{\color{red}16} \otimes \{\mathcal{R}^{(i)}\}_p \right) \cap \{\mathcal{R}^{(j)}\}_{p+1} \right] \quad ,$$

$$(8.32) \qquad c^{-(i)}_{\{\mathcal{R}^{(j)}\}_{p-1}} \;=\; \mathcal{F}_2 \left[\left(\boxed{\overline{16}} \otimes \{\mathcal{R}^{(j)}\}_{p-1} \right) \cap \{\mathcal{R}^{(i)}\}_p \right] \quad ,$$

$$(8.33) \qquad c^{+(j)}_{\{\mathcal{R}^{(i)}\}_{p+1}} \;=\; \mathcal{F}_3 \left[\left(\boxed{\overline{16}} \otimes \{\mathcal{R}^{(j)}\}_{p} \right) \cap \{\mathcal{R}^{(i)}\}_{p+1} \right] \quad ,$$

$$(8.34) \qquad c^{-(j)}_{\{\mathcal{R}^{(i)}\}_{p-1}} \;=\; \mathcal{F}_4 \left[\left(\boxed{\overline{16}} \otimes \{\mathcal{R}^{(i)}\}_{p-1} \right) \cap \{\mathcal{R}^{(j)}\}_{p} \right] \quad ,$$

where $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$, and $\mathcal{F}_4$, are functions. All of these functions have the property that if the intersections indicated as their respective arguments vanish, then the functions output the value of zero. The functions $\mathcal{F}_1$ and, $\mathcal{F}_3$ yield outputs of the value one if their respective intersections are non-vanishing. The functions $\mathcal{F}_2$ and $\mathcal{F}_4$ yield outputs of the value of 0 or 1 if their respective intersections are non-vanishing. When the value of the c-coefficient is 0, there is no link between those particular two nodes. When it is 1, there is a link. For these the intersection principle can only tell us which links must be absent. However, the appearance of the links in the adinkra does not *necessarily* imply the corresponding normalization coefficients have to be non-vanishing.

8.4. Adynkrafields, expansion basis change, and superfields

The discussion in this chapter also points to a relation between the concepts in the adinkras, adynkrafields approach, and traditional description supermultiplets in terms of superfields.

Let us construct a quantity $\mathcal{K}$ from a superfield that can be viewed as an analog to $\mathcal{G}$. We can "strip" a scalar superfield of all of its field components which suggests the construct

$$(8.35) \qquad \mathcal{K} \;=\; 1 \,\oplus\, \theta^\alpha \,\oplus\, \bigoplus_{p=2}^{16} \theta^{\alpha_1} \cdots \theta^{\alpha_p} \quad .$$

Up to normalization factors it can be argued, initially, that the two expressions $\mathcal{G}$ and $\mathcal{K}$ are related to one another via a change of basis $\theta^\alpha \to \ell\,\boxed{}$. This transformation is consistent if there is an understanding that the "red box" actually carries an "invisible" spinor index. We have used this convention in writing (8.3), (8.5), and (8.25). This is also an assumption implicitly used through out this chapter. The "blue boxes" carry invisible vectorial indices, and the "red boxes" carry invisible spinorial indices.

The exponential of the level parameter ℓ plays an important role. It is seen that in (8.3) this parameter tracks the level of the YT's as they appear in the higher dimensional adynkra. So the exponents range from 0 to 16. On the other hand, within the equation (8.25) which applies to only one dimensional valise systems, the exponent of ℓ only takes on values 0 and 1. Within the context of 1D, $N = 16$ theories, there are many possible values of this exponent. The exponents in (8.25) correspond to a two-level adinkra while the ones in (8.3) correspond to a full sixteen-level adinkra in 1D if we set $x^0 = \tau$ and all spatial components $x^i = 0$. There are many other choices for the values of the exponents of ℓ as explored in the work of [43] in one class of examples.

In the context adynkras and adinkras, the exponents of the level parameter play another important role. The exponents control the engineering dimension of the fields that follow in the expansion in (8.3) and (8.25). So for example, the fact that all the bosons in (8.25) are associative with $(\ell)^0$ implies all the bosons possess the same engineering dimensions. Similarly, the fact that all the fermions in (8.25) are associate with $(\ell)^1$ implies all the fermions possess the same engineering dimensions. However, the engineering dimensions of the fermions differs by a unit of $(\text{mass})^{-\frac{1}{2}}$ from the engineering dimensions of the bosons. This follows from the interpretation of adynkrafields arising from superfields via the change of basis $\theta^\alpha \to \ell \,\boxed{}$.

There is one implication about the substitution suggestion as the θ-variable is anti-commuting. This demands that the product $\ell \,\boxed{}$ should also be anti-commuting. The most natural way to do this is to assume the anti-commutivity of the red box.

Finally, the results earlier in this section, may be combined with the result in (7.26) to gain an insight into how this formalism works. We first introduce an even more condense notation by writing the "blue spacetime derivative" $\partial = \boxed{\bullet}\!\!-\!\partial$. Now some samples of component actions can be presented in the simplest cases of actions for the spin-0, spin-1/2, and spin-1 fields from (7.26) as illustrations. The respective actions are simply

$$(8.36) \qquad \mathcal{S}_{\{0\}} = \int d^{10}x \; \tfrac{1}{2} \left\langle \partial\, \phi(x) \,\middle|\, \partial\, \phi(x) \right\rangle \;,$$

$$(8.37) \qquad \mathcal{S}_{\boxed{16}} = \int d^{10}x \; i\, \tfrac{1}{2} \left\langle \lambda_{\boxed{16}}(x) \,\middle|\, \partial \lambda_{\boxed{16}}(x) \right\rangle \;,$$

$$(8.38) \qquad \mathcal{S}_{\boxed{16}} = \int d^{10}x \; i\, \tfrac{1}{2} \left\langle \lambda_{\boxed{16}}(x) \,\middle|\, \partial \lambda_{\boxed{16}}(x) \right\rangle \;,$$

$$(8.39) \qquad \mathcal{S}_{\{1\}} = \int d^{10}x \; \tfrac{1}{4} \left\langle \partial \wedge A_{\boxed{}}(x) \,\middle|\, \partial \wedge A_{\boxed{}}(x) \right\rangle \;,$$

where notation $\langle \, \rangle$ indicates that intermediate calculations are to be done in the space of the YT's and after these are completely evaluated, only the coefficient of the singlet Tableau is retained. We are thus advocating an effectively index-free notation until the final stage of the derivation of results. As Tableaux and Dynkin Labels are interchangeable, this is the orgin of the gain in calculational efficiency based on the used of adynkrafields.

9. Conclusion

In this work we have shown all the steps that allow one to begin with an adynkra of the 10D, $\mathcal{N} = 1$ scalar superfield and apply a well defined set of rules to "tease" from this starting point and finally obtain the field variables (together with their irreducibility conditions) for which the Dynkin Labels provide descriptions.

There remain a few more steps before one obtains a complete component-level description of 10D, $\mathcal{N} = 1$ Nordström supergravity theory. These include:

(a.) the use of the adynkra to provide a starting point for ansatzë for the component level supersymmetry variations on each component field, and

(b.) substitute all of these results in the expressions that appear in [18] which relate 10D, $\mathcal{N} = 1$ Nordström supergeometry to the 10D, $\mathcal{N} = 1$ scalar superfield.

It should be noted the adynkra plays a role in efficiency in part (a.). Namely only representations connected by links in the adynkra are allowed to appear in the component level supersymmetry variations. This together with Lorentz invariance fixes (up to a set of constants) the form of these variations. These final constants are fixed by the condition of closure of the SUSY algebra. Using modern IT applications, it should be possible to completely fix this final set of constants.

However, we should remind the reader that even if this is all explicitly carried out, one still has a reducible construction. That is a separate problem needing further investigation of the properties of the quantities $\widetilde{\mathcal{G}}$ or $\mathcal{G}$.

We believe the explicit presentation in this work should be convincing to the skeptical reader that there exists a well-defined set of steps that starts with the 10D, $\mathcal{N} = 1$ adynkra shown in Figure 1 and leads to the complete component field description given in Chapter 7. To our knowledge, this is

the first such completely explicit presentation at the component level in the literature.

To tie these results with the corresponding geometrical ones discussed in [18], we note that $\Phi(x)$ at Level-0 is the scalar graviton, while $\Psi_\alpha(x)$ at Level-1 is the non-conformal part of the gravitino. Finally, the component field $\Phi_{\{\underline{a_1b_1c_1}\}}(x)$ corresponds to the lowest component of a superfield so that $\Phi_{\{\underline{a_1b_1c_1}\}}(x) = G_{\{\underline{a_1b_1c_1}\}}(x) \propto (\sigma_{\underline{a_1b_1c_1}})_{\alpha\beta} \left(\mathrm{D}^\alpha \mathrm{D}^\beta \mathcal{V}\right)|$. From Eq (5.11) and Eq. (5.14) in the work of [18], we know this field $G_{\{\underline{a_1b_1c_1}\}}$ is the lowest component of a quantity that appears in the supertorsion $T_{\alpha\underline{b}}{}^\gamma$ and the supercurvature $R_{\alpha\beta}{}^{\underline{cd}}$. All the remaining component fields seen at Levels 3 - 16 occur as the lowest components of superfields obtained by applications of spinor derivatives of orders 1 - 14 to $G_{\{\underline{a_1b_1c_1}\}}$ in the supertorsion $T_{\alpha\underline{b}}{}^\gamma$ and the supercurvature $R_{\alpha\beta}{}^{\underline{cd}}$.

For the skeptic, we present this as the most explicit evidence to date that, at least with respect to a supergeometrical formulation, there exists a well-defined theory of 10D, $\mathcal{N} = 1$ Nordström supergravity expressed in terms of the fifteen bosonic and twelve fermionic component fields exhibited in Chapter 7.

In Chapter 4, we have checked the consistency of branching rules of $\mathfrak{su}(10) \supset \mathfrak{so}(10)$ as well as the Weyl dimension formula, the $\mathfrak{su}(10)$ algebra hook rule, and our graphical tying rules for all BYTs up to 3-columns. We are thus able to show our graphical rules for branching rules and the dimensionality of irreps actually have a substantial support. So we cast this into the form of a conjecture.

> **Conjecture:** *The calculation of the branching rules for general $\mathfrak{su}(N) \supset \mathfrak{so}(N)$ where $A_{N-1} \supset D_{N/2}$ for even N, or $A_{N-1} \supset B_{(N-1)/2}$ for odd N, may be found by using the hook rule and the application of the tying rules for that irrep's Young Tableau in $\mathfrak{su}(N)$, if that Young Tableau contains less than or equal to three columns.*

However, this still is not a replacement for a rigorous mathematical proof.

In this work, we have presented preliminary evidence for the existence of adynkrafields which have the potential to replace superfields as tools for the study of dynamical systems that realize supersymmetry. There are still some points about this proposal that remain unclear and will require further study in the future. One obvious consequence of the introduction of adynkrafields is their multiplication rules, following as a result of applying tensor product

rules to the YT's that appear in the expansions of adinkrafields. This observation must have further implications for obtaining a deeper understanding of superspace integration theory.

Finally, in this work we have introduced an essentially coordinate-free "generator of supersymmetry multiplets" in the form of the operator $\mathcal{G}$ that appears in (8.2). It operates on the space of fields $\{\mathcal{F}\}$ introduced in (7.25) to produce scalar supermultiplets via an "inner product" $\mathcal{G} \cdot \{\mathcal{F}\}$. Tensoring $\mathcal{G}$ with any Young Tableaux λ suitable to describe an irrep in the space of fields $\{\mathcal{F}\}$ produces higher representation supermultiplets by tensoring first and then applying the inner product, i.e. $(\lambda \mathcal{G}) \cdot \{\mathcal{F}\}$. It should also be clear that a very similar argument can be made for Salam-Strathdee superfields. Namely a scalar superfield may be regarded as an "inner product" $\mathcal{K} \cdot \{\mathcal{F}\}$.

With respect to supersymmetry, the operator $\mathcal{G}$ is reducible. Understanding how to decompose it remains an unsolved, the one great unsolved, problem of supersymmetry representation theory.

> *"By believing passionately in something that still does not exist, we create it."*

— Franz Kafka

Appendix A. More discussion of index notation

In Section 3.3, we introduced notational conventions to express the index structure which corresponds to the irreducible bosonic Young Tableaux. The general expression was shown in Figure 2. In this appendix, we will present more examples of this notation.

First, recall the general expression of the notation.

(A.1)

$$[p,0,0,0,0] + [0,q,0,0,0] + [0,0,r,0,0] + [0,0,0,s,s] + \begin{array}{l}[0,0,0,0,2t]\text{ with self-duality}\\[0,0,0,2t,0]\text{ with antiself-duality}\end{array} = \begin{array}{l}[p,q,r,s,s+2t]\text{ with self-duality}\\[p,q,r,s+2t,s]\text{ with antiself-duality}\end{array}$$

We start with some simple examples where only $p \neq 0$.

(A.2) $\{\underline{a}_1\} \;\equiv\; \boxed{\underline{a}_1}_{\mathrm{IR}} \;\equiv\; [1,0,0,0,0]$,

(A.3) $\{\underline{a}_1,\underline{a}_2\} \;\equiv\; \boxed{\underline{a}_1\,\underline{a}_2}_{\mathrm{IR}} \;\equiv\; [2,0,0,0,0]$,

(A.4) $\{\underline{a}_1,\underline{a}_2,\underline{a}_3\} \;\equiv\; \boxed{\underline{a}_1\,\underline{a}_2\,\underline{a}_3}_{\mathrm{IR}} \;\equiv\; [3,0,0,0,0]$,

(A.5) $\{\underline{a}_1,\underline{a}_2,\underline{a}_3,\underline{a}_4\} \;\equiv\; \boxed{\underline{a}_1\,\underline{a}_2\,\underline{a}_3\,\underline{a}_4}_{\mathrm{IR}} \;\equiv\; [4,0,0,0,0]$.

Then we turn to some examples where $t = 0$, which means we don't need to consider self duality.

(A.6)

$\{\underline{a}_1\underline{b}_1\} \;\equiv\; \begin{array}{|c|}\hline \underline{a}_1\\\hline \underline{b}_1\\\hline\end{array}_{\mathrm{IR}} \;\equiv\; [0,1,0,0,0]$,

(A.7)

$\{\underline{a}_2|\underline{a}_1\underline{b}_1\} \;\equiv\; \begin{array}{|c|c|}\hline \underline{a}_1 & \underline{a}_2\\\hline \underline{b}_1\\\cline{1-1}\end{array}_{\mathrm{IR}} \;\equiv\; [1,1,0,0,0]$,

(A.8)

$\{\underline{a}_1\underline{b}_1,\underline{a}_2\underline{b}_2\} \;\equiv\; \begin{array}{|c|c|}\hline \underline{a}_1 & \underline{a}_2\\\hline \underline{b}_1 & \underline{b}_2\\\hline\end{array}_{\mathrm{IR}} \;\equiv\; [0,2,0,0,0]$,

(A.9)

$$\{\underline{a}_1\underline{b}_1\underline{c}_1\} \equiv \begin{array}{|c|} \hline a_1 \\ \hline b_1 \\ \hline c_1 \\ \hline \end{array}_{\text{IR}} \equiv [0,0,1,0,0] \ ,$$

(A.10)

$$\{\underline{a}_2|\underline{a}_1\underline{b}_1\underline{c}_1\} \equiv \begin{array}{|c|c|} \hline a_1 & a_2 \\ \hline b_1 \\ \cline{1-1} c_1 \\ \cline{1-1} \end{array}_{\text{IR}} \equiv [1,0,1,0,0] \ ,$$

(A.11)

$$\{\underline{a}_2,\underline{a}_3|\underline{a}_1\underline{b}_1\underline{c}_1\} \equiv \begin{array}{|c|c|c|} \hline a_1 & a_2 & a_3 \\ \hline b_1 \\ \cline{1-1} c_1 \\ \cline{1-1} \end{array}_{\text{IR}} \equiv [2,0,1,0,0] \ ,$$

(A.12)

$$\{\underline{a}_1\underline{b}_1\underline{c}_1,\underline{a}_2\underline{b}_2\underline{c}_2\} \equiv \begin{array}{|c|c|} \hline a_1 & a_2 \\ \hline b_1 & b_2 \\ \hline c_1 & c_2 \\ \hline \end{array}_{\text{IR}} \equiv [0,0,2,0,0] \ ,$$

(A.13)

$$\{\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1,\underline{a}_2\underline{b}_2\underline{c}_2\underline{d}_2,\underline{a}_3\underline{b}_3\underline{c}_3\underline{d}_3,\underline{a}_4\underline{b}_4\underline{c}_4\underline{d}_4\} \equiv \begin{array}{|c|c|c|c|} \hline a_1 & a_2 & a_3 & a_4 \\ \hline b_1 & b_2 & b_3 & b_4 \\ \hline c_1 & c_2 & c_3 & c_4 \\ \hline d_1 & d_2 & d_3 & d_4 \\ \hline \end{array}_{\text{IR}} \equiv [0,0,0,4,4] \ .$$

Finally, some examples where we have to take self duality into account are presented as below.

(A.14) $\quad \{\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^+ \equiv \begin{array}{|c|} \hline a_1 \\ \hline b_1 \\ \hline c_1 \\ \hline d_1 \\ \hline e_1 \\ \hline \end{array}_{\text{IR},+} \equiv [0,0,0,0,2] \ ,$

(A.15) $\quad \{\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^- \equiv \begin{array}{|c|} \hline a_1 \\ \hline b_1 \\ \hline c_1 \\ \hline d_1 \\ \hline e_1 \\ \hline \end{array}_{\text{IR},-} \equiv [0,0,0,2,0] \ ,$

$$(A.16) \qquad \{\underline{a}_2|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^+ \quad \equiv \quad \begin{array}{|c|c|} \hline a_1 & a_2 \\ \hline b_1 \\ \cline{1-1} c_1 \\ \cline{1-1} d_1 \\ \cline{1-1} e_1 \\ \cline{1-1} \end{array}_{\text{IR},+} \quad \equiv \quad [1,0,0,0,2] \quad ,$$

$$(A.17) \qquad \{\underline{a}_2|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^- \quad \equiv \quad \begin{array}{|c|c|} \hline a_1 & a_2 \\ \hline b_1 \\ \cline{1-1} c_1 \\ \cline{1-1} d_1 \\ \cline{1-1} e_1 \\ \cline{1-1} \end{array}_{\text{IR},-} \quad \equiv \quad [1,0,0,2,0] \quad ,$$

$$(A.18) \qquad \{\underline{a}_2\underline{b}_2|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^+ \quad \equiv \quad \begin{array}{|c|c|} \hline a_1 & a_2 \\ \hline b_1 & b_2 \\ \hline c_1 \\ \cline{1-1} d_1 \\ \cline{1-1} e_1 \\ \cline{1-1} \end{array}_{\text{IR},+} \quad \equiv \quad [0,1,0,0,2] \quad ,$$

$$(A.19) \qquad \{\underline{a}_2\underline{b}_2|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^- \quad \equiv \quad \begin{array}{|c|c|} \hline a_1 & a_2 \\ \hline b_1 & b_2 \\ \hline c_1 \\ \cline{1-1} d_1 \\ \cline{1-1} e_1 \\ \cline{1-1} \end{array}_{\text{IR},-} \quad \equiv \quad [0,1,0,2,0] \quad .$$

Appendix B. Explicit examples of spinorial irrep dimension formulas

In this appendix, we list explicit examples of spinorial irrep dimension formulas for completely antisymmetric, completely symmetric, two-equal-column and two-unequal-column BYTs attached with a {16} respectively. These are the types of spinorial irreps that appear in the 10D, $\mathcal{N} = 1$ scalar superfield.

In the following sections, we list the irreps in Dynkin Label, mixed YT notation and dimensions on the left, and how we calculate the dimensions graphically and numerically on the right. Note that on the right, we omit all the "dim" notation for compact presentation. Hence the "×" just means multiplying the dimensions, but *not* any sort of tensor product or direct product; and the "−" just means subtracting the corresponding dimensions, but *not* any sort of complement.

B.1. Completely antisymmetric BYTs attached with {16}

(B.1) $[1,0,0,0,1] = \boxed{}\boxed{}_{\text{IR}}$ $\left(\boxed{}_{\text{IR}} - \cdot \right) \times \boxed{}$

$= \{\overline{144}\}$ $(\quad 10 \quad - \quad 1 \quad) \quad \times \quad 16$

(B.2) $[0,1,0,0,1] = \square_{\text{IR}}$ $\left(\square_{\text{IR}} - \square_{\text{IR}} \right) \times \square$

$= \{560\}$ $(\quad 45 \quad - \quad 10 \quad) \quad \times \quad 16$

(B.3) $[0,0,1,0,1] = \square_{\text{IR}}$ $\left(\square_{\text{IR}} - \square_{\text{IR}} \right) \times \square$

$= \{\overline{1200}\}$ $(\quad 120 \quad - \quad 45 \quad) \quad \times \quad 16$

(B.4) $[0,0,0,1,2] = \square_{\text{IR}}$ $\left(\square_{\text{IR}} - \square_{\text{IR}} \right) \times \square$

$= \{1440\}$ $(\quad 210 \quad - \quad 120 \quad) \quad \times \quad 16$

(B.5) $[0,0,0,0,3] = \square_{\text{IR},+}$ $\left(\square_{\text{IR}} - \square_{\text{IR}} \right) \times \square$

$= \{\overline{672}\}$ $(\ (126 + 126) \ - \ 210 \ ) \ \times \ 16$

B.2. Completely symmetric BYTs attached with {16}

(B.6)

$[2,0,0,0,1] = \square_{\text{IR}}$ $\left(\square_{\text{IR}} - \square_{\text{IR}} + \cdot \right) \times \square$

$= \{720\}$ $(\quad 54 \quad - \quad 10 \quad + \quad 1 \ ) \ \times \ 16$

(B.7)

$[3,0,0,0,1] = \square_{\text{IR}}$ $\left(\square_{\text{IR}} - \square_{\text{IR}} + \square_{\text{IR}} - \cdot \right) \times \square$

$= \{\overline{2640}\}$ $(\quad 210 \quad - \quad 54 \quad + \quad 10 \quad - \quad 1 \ ) \ \times \ 16$

(B.8)

$$[4,0,0,0,1] \;=\; \boxed{}_{\text{IR}} \quad \left(\boxed{}_{\text{IR}} - \boxed{}_{\text{IR}} + \boxed{}_{\text{IR}} - \boxed{}_{\text{IR}} + \cdot \right) \times \boxed{}$$

$$= \{7920\} \qquad (\quad 660 \quad - \quad 210 \quad + \quad 54 \quad - \quad 10 \; + \; 1 \;) \times 16$$

B.3. Two-equal-column BYTs attached with {16}

(B.9)

$$[0,2,0,0,1] \;=\; \boxed{}_{\text{IR}} \qquad \left(\boxed{}_{\text{IR}} - \boxed{}_{\text{IR}} + \boxed{}_{\text{IR}} \right) \times \boxed{}$$

$$= \{8064\} \qquad (\quad 770 \quad - \quad 320 \quad + \quad 54 \quad) \times 16$$

(B.10)

$$[0,0,2,0,1] \;=\; \boxed{}_{\text{IR}} \qquad \left(\boxed{}_{\text{IR}} - \boxed{}_{\text{IR}} + \boxed{}_{\text{IR}} \right) \times \boxed{}$$

$$= \{30800\} \qquad (\quad 4125 \quad - \quad 2970 \quad + \quad 770 \quad) \times 16$$

(B.11)

$$[0,0,0,2,3] \;=\; \boxed{}_{\text{IR}} \qquad \left(\boxed{}_{\text{IR}} - \boxed{}_{\text{IR}} + \boxed{}_{\text{IR}} \right) \times \boxed{}$$

$$= \{39600\} \qquad (\quad 8910 \quad - \quad 10560 \quad + \quad 4125 \quad) \times 16$$

(B.12)

$$[0,0,0,0,5] \;=\; \boxed{}_{\text{IR},+} \qquad \left(\boxed{}_{\text{IR}} - \boxed{}_{\text{IR}} + \boxed{}_{\text{IR}} \right) \times \boxed{}$$

$$= \{9504\} \qquad (\, (2772 + 2772) \; - \; (6930 + 6930) \; + \; 8910 \,) \times 16$$

B.4. Two-unequal-column BYTs attached with {16}

(B.13)

$$[1,1,0,0,1] = \text{(diagram)}_{\text{IR}}$$
$$= \{\overline{3696}\}$$

$$\left(\text{(diagram)}_{\text{IR}} - \text{(diagram)}_{\text{IR}} - \text{(diagram)}_{\text{IR}} + \text{(diagram)}_{\text{IR}} \right) \times \text{(diagram)}$$
$$(\quad 320 \quad - \quad 54 \quad - \quad 45 \quad + \quad 10 \quad) \times 16$$

(B.14)

$$[0,1,1,0,1] = \text{(diagram)}_{\text{IR}}$$
$$= \{\overline{25200}\}$$

$$\left(\text{(diagram)}_{\text{IR}} - \text{(diagram)}_{\text{IR}} - \text{(diagram)}_{\text{IR}} + \text{(diagram)}_{\text{IR}} \right) \times \text{(diagram)}$$
$$(\quad 2970 \quad - \quad 770 \quad - \quad 945 \quad + \quad 320 \quad) \times 16$$

(B.15)

$$[0,0,1,0,3] = \text{(diagram)}_{\text{IR}}$$
$$= \{29568\}$$

$$\left(\text{(diagram)}_{\text{IR}} - \text{(diagram)}_{\text{IR}} - \text{(diagram)}_{\text{IR}} + \text{(diagram)}_{\text{IR}} \right) \times \text{(diagram)}$$
$$(\quad (6930 + 6930) \quad - \quad 10560 \quad - \quad (3696 + 3696) \quad + \quad 5940 \quad) \times 16$$

Appendix C. Spinorial irreps of $\mathfrak{so}(10)$ in field theory notation by tensor products

In Chapter 6, we described the tensor product rule for a general bosonic irrep with the basic spinorial irrep, and explained how the Schur function series Q serves as an inverse of the Schur function series P by verifying the tensor product rule from the SYT dimension rule. In this appendix, we will turn to the spinorial irreps that appear in the 10D, $\mathcal{N} = 1$ scalar superfield, i.e.

(C.1)

In this appendix, we plan to achieve two goals:

(a.) provide some explicit examples of tensor product rules; and

(b.) show that Q is the inverse of P through these examples by obtaining the formulas in Appendix B from these tensor product rules.

Note that (b.) is the converse of what we did in Section 6.

For the first SYT in (C.1), the relevant tensor product rule is

(C.2)
$$\cdot \ \otimes \ \square \ = \ \boxed{16}$$
$$\{1\} \otimes \{16\} = \{16\}$$
,

so that there is no irreducible condition, and we know that it translates to index notations as

(C.3) $\{16\} \ = \ \Psi^\alpha$,

(C.4) $\{\overline{16}\} \ = \ \Psi_\alpha$.

For the second and the third SYTs in (C.1), the relevant tensor product rules are those totally antisymmetric BYTs tensored with a basic spinorial irrep. They are listed as follows.

(C.5)

$$\square_{\text{IR}} \ \otimes \ \square \ = \ \square\,\boxed{16}_{\text{IR}} \ \oplus \ \boxed{\overline{16}}$$
$$\{10\} \otimes \{16\} = \{144\} \ \oplus \{16\}$$
,

(C.6)

$$\begin{smallmatrix}\square\\\square\end{smallmatrix}_{\text{IR}} \ \otimes \ \square \ = \ \begin{smallmatrix}\square\,\boxed{16}\\\square\end{smallmatrix}_{\text{IR}} \ \oplus \ \square\,\boxed{\overline{16}}_{\text{IR}} \ \oplus \ \boxed{16}$$
$$\{45\} \otimes \{16\} = \{560\} \ \oplus \{144\} \ \oplus \{16\}$$
,

(C.7)

$$\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}_{\text{IR}} \ \otimes \ \square \ = \ \begin{smallmatrix}\square\,\boxed{16}\\\square\\\square\end{smallmatrix}_{\text{IR}} \ \oplus \ \begin{smallmatrix}\square\,\boxed{\overline{16}}\\\square\end{smallmatrix}_{\text{IR}} \ \oplus \ \square\,\boxed{16}_{\text{IR}} \ \oplus \ \boxed{\overline{16}}$$
$$\{120\} \otimes \{16\} = \{\overline{1200}\} \ \oplus \{560\} \ \oplus \{\overline{144}\} \ \oplus \{\overline{16}\}$$
,

(C.8)

$$\begin{smallmatrix}\square\\\square\\\square\\\square\end{smallmatrix}_{\text{IR}} \ \otimes \ \square \ = \ \begin{smallmatrix}\square\,\boxed{16}\\\square\\\square\\\square\end{smallmatrix}_{\text{IR}} \ \oplus \ \begin{smallmatrix}\square\,\boxed{\overline{16}}\\\square\\\square\end{smallmatrix}_{\text{IR}} \ \oplus \ \begin{smallmatrix}\square\,\boxed{16}\\\square\end{smallmatrix}_{\text{IR}} \ \oplus \ \square\,\boxed{\overline{16}}_{\text{IR}} \ \oplus \ \boxed{16}$$
$$\{210\} \otimes \{16\} = \{1440\} \ \oplus \{1200\} \ \oplus \{560\} \ \oplus \{144\} \ \oplus \{16\}$$
,

(C.9)

$$(\{126\} \oplus \{\overline{126}\}) \otimes \{16\} = \{\overline{672}\} \oplus \{1440\} \oplus \{1200\} \oplus \{560\} \oplus \{144\} \oplus \{16\} \quad ,$$

where Equation (C.9) can be split into

(C.10)

$$\{\overline{126}\} \otimes \{16\} = \{\overline{672}\} \oplus \{1200\} \oplus \{144\} \quad ,$$

(C.11)

$$\{126\} \otimes \{16\} = \{1440\} \oplus \{560\} \oplus \{16\} \quad .$$

From the above tensor product rules, we obtain

(C.12)

$$\{560\} = \{45\} \otimes \{16\} - \{10\} \otimes \{\overline{16}\} \quad ,$$

(C.13)

$$\{\overline{672}\} = (\{126\} \oplus \{\overline{126}\}) \otimes \{16\} - \{210\} \otimes \{\overline{16}\} \quad ,$$

and their conjugates. The "−" here denotes the removal of the terms in the direct sum. Note that the dimensions obtained from these two equations agree exactly with Equations (B.2) and (B.5).

For the forth SYT in (C.1), the relevant tensor product rules are Equations (C.2) and (C.5), and the totally symmetric BYTs tensored with a basic spinorial irrep listed as follows.

(C.14)
$$\{54\} \otimes \{16\} = \{720\} \oplus \{144\}$$

(C.15)
$$\{210'\} \otimes \{16\} = \{2640\} \oplus \{720\}$$

From these tensor product rules we derive

(C.16)
$$\{2640\} = \{210'\} \otimes \{16\} - \{54\} \otimes \{\overline{16}\} \oplus \{10\} \otimes \{16\} - \{1\} \otimes \{\overline{16}\}$$

which is consistent with (B.7).

For the last two SYTs in (C.1), the relevant tensor product rules are Equations (C.5), (C.6), (C.7), (C.14) and the two following equations,

(C.17)
$$\{320\} \otimes \{16\} = \{\overline{3696}\} \oplus \{\overline{560}\} \oplus \{\overline{720}\} \oplus \{\overline{144}\}$$

(C.18)
$$\{945\} \otimes \{16\} = \{8800\} \oplus \{1200\} \oplus \{3696\} \oplus \{560\} \oplus \{720\} \oplus \{144\}$$

Then we have

(C.19)
$$\{\overline{3696}\} = \{320\} \otimes \{16\} - \{45\} \otimes \{\overline{16}\} - \{54\} \otimes \{\overline{16}\} \oplus \{10\} \otimes \{16\}$$

(C.20)
$$\{8800\} = \{945\} \otimes \{16\} - \{120\} \otimes \{\overline{16}\} - \{320\} \otimes \{\overline{16}\} \oplus \{45\} \otimes \{16\}$$

where Equation (C.19) agrees with Equation (B.13).

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Brown Theoretical Physics Center
Box S, 340 Brook Street, Barus Hall
Providence, RI 02912, USA
and Department of Physics, Brown University
Box 1843, 182 Hope Street, Barus & Holley
Providence, RI 02912, USA
E-mail address: `sylvester_gates@brown.edu`
E-mail address: `yangrui_hu@brown.edu`
E-mail address: `sze_ning_mak@brown.edu`