Zitterbewegung in Quantum Mechanics – a research program

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Spacetime Algebra (STA) provides unified, matrix-free spinor methods for rotational dynamics in classical theory as well as quantum mechanics. That makes it an ideal tool for studying particle models of *zitterbewegung* and using them to study zitterbewegung in the Dirac theory. This paper develops a self-contained dynamical model of the electron as a lightlike particle with helical zitterbewegung and electromagnetic interactions. It attributes to the electron an electric dipole moment oscillating with ultrahigh frequency, and the possibility of observing this directly as a resonance in electron channeling is analyzed in detail. A modification of the Dirac equation is suggested to incorporate the oscillating dipole moment. That enables extension of the Dirac equation to incorporate electroweak interactions in a new way.

Keywords: zitterbewegung, geometric algebra, electron channeling, de Broglie frequency

I. INTRODUCTION

This paper continues a research program investigating implications of the *Real Dirac Equation* for the interpretation and extension of quantum mechanics. Details of the program have been reviewed elsewhere [1–3], so it suffices here to state the main ideas and conclusions to set the stage for the present study.

The program began with a reformulation of the Dirac equation in terms of *Spacetime Algebra* (Section II), which revealed geometric structure that is suppressed in the standard matrix version. In particular, it revealed that the generator of phase and electromagnetic gauge transformations is a spacelike bivector specified by electron spin. In other words, spin and phase are inseparably related — spin is not simply an add-on, but an essential feature of quantum mechanics. However, physical implications of this fact depend critically on relations of the Dirac wave function to physical observables, which are not specified by the Dirac equation itself. That started the present research program to investigate various possibilities.

A standard observable in Dirac theory is the Dirac current, which doubles as a probability current and a charge current. However, this does not account for the magnetic moment of the electron, which many investigators conjecture is due to a circulation of charge. But what is the nature of this circulation? After a lengthy analysis of the Dirac equation Bohm and Hiley conclude [4]: "the electron must still be regarded as a simple point particle whose only intrinsic property is its position." Under this assumption, spin and phase must be kinematical features of electron motion. The charge circulation that generates the magnetic moment can then be identified with the *zitterbewegung* of Schroedinger [5].

This raises the central question of the present research: Is the zitterbewegung, so construed, a real physical phenomenon, or is it merely a colorful metaphor? Although this question was motivated by structural features of the Dirac equation, it cannot be answered without attributing substructure to electron motion that is not specified by standard Dirac theory.

The main purpose of this paper is to formulate and study a well-defined particle model of the electron with *spin and zitterbewegung dynamics* motivated by the Dirac equation. Since the term *zitterbewegung* is quite a mouthful, I often abbreviate it to *zitter*, especially when it is used as an adjective.

We study the structure of the *zitter model* in considerable detail with the aim of identifying new experimental implications. The main conclusion is that the electron is the seat of a rapidly rotating electric dipole moment fluctuating with the zitter frequency of Schroedinger. As this frequency is so rapid, it is observable only under resonance conditions. It is argued that many familiar quantum mechanical effects may be attributable to zitter resonance. Moreover, the new possibility of observing zitter directly as a resonance in electron channeling is analyzed in detail, because the prospects of crucial experimental tests are very promising.

The relation of the zitter particle model to the Dirac equation is also studied. The main conclusion is that, though zitter oscillations are inherent in the Dirac equation, they will not be manifested as an oscillating electric dipole without altering the definition of charge current. A simple modification of the Dirac equation to incorporate the altered definition is proposed. Remarkably, that opens the door for incorporating electroweak interactions into the Dirac equation in a novel way.

In conclusion, the relation of the zitter particle model to the Dirac equation can be considered from two different perspectives. On the one hand, it can be regarded as a "quasiclassical" approximation that embodies structural features of the Dirac equation in a convenient form for analysis. On the other hand, it can be regarded as formulating fundamental properties of the electron that are manifested in the Dirac equation in some kind of average form. The choice of perspective is left to the reader.

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II. SPACETIME ALGEBRA

Spacetime algebra is thoroughly expounded elsewhere [1], so a brief description is sufficient here, mainly to define terms.

We represent Minkowski *spacetime* as a real 4-dimensional vector space $\mathcal{M}^4$. The two properties of scalar multiplication and vector addition in $\mathcal{M}^4$ provide only a partial specification of spacetime geometry. To complete the specification we introduce an associative *geometric product* among vectors $a, b, c, \ldots$ with the property that the square of any vector is a (real) scalar. Thus for any vector a we can write

$$a^2 = aa = \epsilon |a|^2 , \tag{1}$$

where ϵ is the *signature* of a and $|a|$ is a (real) positive scalar. As usual, we say that a is *timelike*, *lightlike* or *spacelike* if its signature is positive ($\epsilon = 1$), null ($\epsilon = 0$), or negative ($\epsilon = -1$). We can specify the signature of $\mathcal{M}^4$ as a whole, by adopting the axioms: (a) $\mathcal{M}^4$ contains at least one timelike vector; and (b) every 2-plane in $\mathcal{M}^4$ contains at least one spacelike vector.

To facilitate applications of STA to physics a few definitions and theorems are needed. From the *geometric product* uv of two vectors it is convenient to define two other products. The *inner product* $u \cdot v$ is defined by

$$u \cdot v = \tfrac{1}{2}(uv + vu) = v \cdot u , \tag{2}$$

while the *outer product* $u \wedge v$ is defined by

$$u \wedge v = \tfrac{1}{2}(uv - vu) = -v \wedge u . \tag{3}$$

The three products are therefore related by

$$uv = u \cdot v + u \wedge v . \tag{4}$$

This can be regarded as a decomposition of the product uv into symmetric and skewsymmetric parts, or alternatively, into scalar and bivector parts.

The inner and outer products can be generalized. One way is to define the outer product along with the notion of *k-vector* iteratively as follows: scalars are defined to be 0-vectors, vectors are 1-vectors, and bivectors, such as $u \wedge v$, are 2-vectors. For a given k-vector K, the integer k is called the *grade* of K. For $k \geq 1$, the outer product of a vector v with a k-vector K is a $(k+1)$-vector defined in terms of the geometric product by

$$v \wedge K = \tfrac{1}{2}(vK + (-1)^k Kv) = (-1)^k K \wedge v . \tag{5}$$

The corresponding inner product is defined by

$$v \cdot K = \tfrac{1}{2}(vK + (-1)^{k+1} Kv) = (-1)^{k+1} K \cdot v , \tag{6}$$

and it can be proved that the result is a $(k-1)$-vector. Adding (5) and (6) we obtain

$$vK = v \cdot K + v \wedge K , \tag{7}$$

which obviously generalizes (4). The important thing about (7), is that it decomposes vK into $(k-1)$-vector and $(k+1)$-vector parts.

By continuing as above, STA as been developed into a complete coordinate-free calculus for spacetime physics [1]. However, to hasten comparison with standard Dirac algebra, we interrupt that approach to introduce coordinates and a basis for the algebra. Let $\{\gamma_\mu;\ 0, 1, 2, 3\}$ be a *right-handed orthonormal frame* of vectors with γ_0 in the forward light cone. In accordance with (2), the components $\mathrm{g}_{\mu\nu}$ of the metric tensor for this frame are given by

$$\mathrm{g}_{\mu\nu} = \gamma_\mu \cdot \gamma_\nu = \tfrac{1}{2}(\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu) . \tag{8}$$

This will be recognized as isomorphic to a famous formula of Dirac's. The difference here is that the γ_μ are vectors rather than matrices. A *coframe* $\{\gamma^\mu\}$ is defined by the usual convention for raising and lowering indices: $\gamma_\mu = \mathrm{g}_{\mu\nu}\gamma^\nu$.

The unit pseudoscalar i for spacetime is related to the frame $\{\gamma_\nu\}$ by the equation

$$i = \gamma_0 \gamma_1 \gamma_2 \gamma_3 = \gamma_0 \wedge \gamma_1 \wedge \gamma_2 \wedge \gamma_3 . \tag{9}$$

It is readily verified from (9) that $i^2 = -1$, and the geometric product of i with any vector is anticommutative.

By multiplication the γ_μ generate a complete basis of k-vectors for STA, consisting of the $2^4 = 16$ linearly independent elements

$$1, \quad \gamma_\mu, \quad \gamma_\mu \wedge \gamma_\nu, \quad \gamma_\mu i, \quad i . \tag{10}$$

Multivectors with even grade compose a subalgebra of the STA generated by the bivectors $\{\boldsymbol{\sigma}_k \equiv \gamma_k \gamma_0; k = 1, 2, 3\}$, so that

$$\boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2 \boldsymbol{\sigma}_3 = \gamma_0 \gamma_1 \gamma_2 \gamma_3 = i. \tag{11}$$

Any multivector can be expressed as a linear combination of these elements.

For example, a bivector F has the expansion

$$F = \tfrac{1}{2}F^{\mu\nu}\gamma_\mu \wedge \gamma_\nu, \tag{12}$$

with its "scalar components" $F^{\mu\nu}$ given by

$$F^{\mu\nu} = \gamma^\mu \cdot F \cdot \gamma^\nu = \gamma^\nu \cdot (\gamma^\mu \cdot F) = (\gamma^\nu \wedge \gamma^\mu) \cdot F . \tag{13}$$

Note that the two inner products in the middle term can be performed in either order, so a parenthesis is not needed; also, we use the usual convention for raising and lowering indices.

Alternatively, one can decompose the bivector into

$$F = \mathbf{E} + i\mathbf{B} = F^{k0}\boldsymbol{\sigma}_k + \tfrac{1}{2}F^{kj}\boldsymbol{\sigma}_j\boldsymbol{\sigma}_k, \tag{14}$$

corresponding to the split of an electromagnetic field into "electric and magnetic parts."

The entire spacetime algebra is obtained from linear combinations of basis k-vectors in (10). A generic element M of the STA, called a *multivector*, can thus be written in the *expanded form*

$$M = \alpha + a + F + bi + \beta i = \sum_{k=0}^{4} \langle M \rangle_k , \tag{15}$$

where α and β are scalars, a and b are vectors, and F is a bivector. This is a decomposition of M into its k-vector parts, with $k = 0, 1, 2, 3, 4$, where $\langle \ldots \rangle_k$ means "k-vector part." Of course, $\langle M \rangle_0 = \alpha$, $\langle M \rangle_1 = a$, $\langle M \rangle_2 = F$, $\langle M \rangle_3 = bi$, $\langle M \rangle_4 = \beta i$.

Computations are facilitated by the operation of *reversion*. For M in the expanded form (15) the *reverse* $\widetilde{M}$ can be defined by

$$\widetilde{M} = \alpha + a - F - bi + \beta i. \tag{16}$$

Note, in particular, the effect of reversion on the various k-vector parts:
$\widetilde{\alpha} = \alpha$, $\quad \widetilde{a} = a$, $\quad \widetilde{F} = -F$, $\quad \widetilde{i} = i$. It is not difficult to prove that

$$(MN)^{\sim} = \widetilde{N}\widetilde{M}, \tag{17}$$

for arbitrary multivectors M and N. For scalar parts $\langle M \rangle \equiv \langle M \rangle_0$, we have

$$\langle \widetilde{M} \rangle = \langle M \rangle \qquad \text{hence} \qquad \langle MN \rangle = \langle NM \rangle. \tag{18}$$

The scalar part in STA corresponds to the trace in Dirac's matrix representation of the algebra.

Besides the inner and outer products defined above, many other products can be defined in terms of the geometric product. We will need the *commutator* product $M \times N$, defined for any M and N by

$$M \times N \equiv \tfrac{1}{2}(MN - NM) = -N \times M. \tag{19}$$

We will make frequent use of the "Jacobi identity" in the form of a "derivation":

$$M \times (N \times P) = (M \times N) \times P + N \times (M \times P). \tag{20}$$

For bivectors S and F we can now expand the geometric product as follows:

$$SF = S \cdot F + S \times F + S \wedge F \tag{21}$$

where $S \times F$ is a bivector and $S \wedge F$ is a pseudoscalar. Note how this differs from the expansion (7).

One great advantage of STA is that it provides coordinate-free representations of both tensors and spinors in the same system. Spinors can be represented as even multivectors serving as algebraic operators. Using reversion, it is easy to prove that every even multivector ψ satisfies

$$\psi\widetilde{\psi} = \rho e^{i\,\beta}. \tag{22}$$

Hence, for $\rho \neq 0$, ψ can be written in the *canonical form*

$$\psi = (\rho e^{i\,\beta})^{\frac{1}{2}}R, \tag{23}$$

where

$$R\widetilde{R} = \widetilde{R}R = 1. \tag{24}$$

Every such R determines a *Lorentz rotation* of a given multivector M:

$$R : M \quad \rightarrow \quad M' = RM\tilde{R}, \tag{25}$$

and every Lorentz rotation can be expressed in this coordinate-free form. Construed as an operator in this sense, the quantity R is called a *rotor* while ψ is called a *spinor*. We shall describe particle kinematics and the Dirac wave function by spinors in this sense.

The set of all rotors form a multiplicative group called the *rotor group* or the *spin group* of spacetime. When $R = R(\tau)$ represents a one parameter family of rotors (hence of Lorentz rotations), "angular velocities" Ω and Ω' are defined by

$$\dot{R} = \tfrac{1}{2}\Omega R = \tfrac{1}{2}R\Omega', \tag{26}$$

or

$$\Omega = 2\dot{R}\widetilde{R} = R\,\Omega'\,\widetilde{R} \qquad \text{and} \qquad \Omega' = 2\widetilde{R}\dot{R}, \tag{27}$$

where the overdot indicates derivative. It follows from (24) and (15) that Ω and Ω' are bivectors.

We represent each spacetime point as a vector $x = x^\mu \gamma_\mu$ with rectangular coordinates $x^\mu = \gamma^\mu \cdot x$. The *vector derivative* with respect to the point x is defined by $\nabla = \partial_x = \gamma^\mu \partial_{x^\mu}$. As ∇ is a vectorial operator, we can use (7) to decompose the derivative of a k-vector field $K = K(x)$ into

$$\nabla K = \nabla \cdot K + \nabla \wedge K, \tag{28}$$

where the terms on the right can be identified, respectively, with the usual *divergence* and *curl* in tensor calculus.

Besides the STA definitions and relations given above, many others can be found in the references. We shall introduce some of them as needed.

III. CLASSICAL PARTICLES WITH SPIN

Classical models of the electron as a point particle with spin were first formulated by Frenkel [6] and Thomas [7], improved by Mathisson [8], and given a definitive form by Weyssenhoff [9]. They have been revisited from time to time by many investigators [10–12], including others to be cited below. They are of interest mainly for the insight they bring to the interpretation of quantum mechanics. But they are also of practical value, for example, in the study of spin precession [1, 13] and tunneling [14–16].

In Weyssenhoff's analysis the models fall into distinct classes, differentiated by the assumption that the electron's spacetime history is timelike in one and lightlike in the other. The timelike case has been studied by many investigators, such as those cited above. Ironically, the lightlike case, which Weyssenhoff regarded as far more interesting, has been generally ignored. Without being aware of his analysis, I arrived at similar conclusions from my study of real Dirac theory. This paper revisits Weyssenhoff theory with new mathematical tools to simplify and extend it.

Our first task is to reformulate Weyssenhoff theory in the language of spacetime algebra (STA) and show how it

is simplified, clarified and extended. A decided advantage of STA is its uniform use of spinors in both classical and quantum theory. For one thing, this makes it easier to relate classical models to solutions of the Dirac equation [1]. Although we are most interested in the lightlike case, we treat the timelike case in enough detail to compare the two cases. Besides, the timelike models can be used for other particles with spin besides electrons, including atoms.

We consider a particle with spacetime *history* $z = z(\tau)$ and *proper velocity* [1]

$$u \equiv \dot{z} = \frac{dz}{d\tau}\,. \tag{29}$$

For the time being, we allow u to be either timelike, with $u^2 = 1$, or lightlike, with $u^2 = 0$, so in either case

$$\dot{u} \cdot u = 0\,. \tag{30}$$

In the timelike case, of course, the parameter τ is taken to be proper time, but in the lightlike case it must be regarded as arbitrary for the time being.

Suppose the particle has *proper momentum* $p = p(\tau)$ and intrinsic angular momentum (*spin*) represented by a bivector $S = S(\tau)$. Equations of motion are obtained from general conservation laws for momentum and angular momentum. However, to formulate these laws correctly it is essential to note that just as p contributes an orbital momentum, S contributes an intrinsic part to the momentum, so we cannot make the usual assumption $p = mu$. Instead, the relation between p and u depends on the dynamics of motion. The same can be said about the relation between S and u, although we shall assume that it is restricted by the *kinematical constraint*

$$S \cdot u = 0\,. \tag{31}$$

This reduces the degrees of freedom in S to three at most.

The noncollinearity of p and u raises a question about how mass should be defined. Without prejudicing the issue, it is convenient to introduce a *dynamical mass* defined by

$$m \equiv p \cdot u\,. \tag{32}$$

This quantity is well defined for both timelike and lightlike particles, so lightlike particles can have mass in this sense. However, the value of this mass obviously depends on the choice of time parameter τ, which, so far, is not well defined in the lightlike case. This is our first hint that mass is intimately related to intrinsic time scaling of electron histories.

Without presupposing a relation between momentum and velocity, *momentum conservation* can be given the general form

$$\dot{p} = f\,, \tag{33}$$

where the *proper force* f describes transfer of momentum through external interactions.

Angular momentum conservation is governed by

$$\dot{S} = u \wedge p + \Gamma\,, \tag{34}$$

where the *proper torque* Γ describes angular momentum transfer through external interactions. To see where (34) comes from, we introduce the *orbital angular momentum* $p \wedge z$ so the *total angular momentum* is

$$J \equiv p \wedge z + S\,. \tag{35}$$

The angular momentum conservation then has the more familiar form

$$\dot{J} = f \wedge z + \Gamma\,, \tag{36}$$

where $f \wedge z$ is the orbital torque. The equivalence of (36) with (34) follows immediately by differentiating (35).

Equation (36) tells us that J is a constant of motion for a free particle with spin. Spin is not separately conserved. According to (35), angular momentum can be exchanged back and forth between orbital and spin parts.

To get well defined equations of motion from the conservation laws we need to specify the interactions. For a particle with charge q and dipole moment $M = M(\tau)$, we use

$$f = qF \cdot u + \nabla F \cdot M \tag{37}$$

where $F = F(z)$ is an applied electromagnetic (bivector) field [aptly called the *Faraday*] and the vector derivative $\nabla = \gamma^\mu \partial_\mu$ operates on F. The first term on the right of (37) is the standard "Lorentz force," while the second is a force of "Stern-Gerlach" type. The latter is accompanied by the torque

$$\Gamma = F \times M\,. \tag{38}$$

The "interaction laws" (37) and (38) must be supplemented by a "constitutive equation" expressing M as a function of S and u. In general M can include both electric and magnetic dipole moments. The general constitutive constraint

$$M \times S = 0 \tag{39}$$

is satisfied by the standard constitutive equation $M = \kappa S$ for a magnetic moment, but also by the more general relation

$$M = \kappa S e^{i\beta}\,. \tag{40}$$

The factor $e^{i\beta}$ is a "duality rotation" relating a magnetic moment $iS\kappa \sin\varphi$ to an electric moment $S\kappa \cos\varphi$. As noted later, the magnetic moment (density) in Dirac theory has this form. In the lightlike case we shall see that there is an alternative way to satisfy (39).

From the specific form of the interaction laws we can prove that the magnitude of the spin is a conserved quantity. First note that (31) implies

$$S \wedge S = 0 \qquad \text{so} \qquad S^2 = S \cdot S\,. \tag{41}$$

This follows from the identity

$$(S \wedge S)u = (S \wedge S) \cdot u = 2S \wedge (S \cdot u) \,.$$

From (34) with (38) we get

$$\tfrac{1}{2}\frac{d}{d\tau}S^2 = S \cdot \dot{S} = S \cdot (u \wedge p) + S \cdot (M \times F) \,.$$

The first term on the right vanishes because

$$S \cdot (u \wedge p) = (S \cdot u) \cdot p = 0 \,,$$

and the second term vanishes by (39) and the identity

$$S \cdot (F \times M) = \langle SFM \rangle = F \cdot (M \times S) \,. \tag{42}$$

Therefore

$$S^2 = -|S|^2 = \text{constant} \,. \tag{43}$$

The negative sign appears because (31) implies that S cannot be a timelike bivector.

Up to this point, our equations apply equally to timelike and lightlike particles. To go further we must consider each of these cases separately. There are three distinct possibilities which have been studied in the past. Most investigators have been attracted by the possibility of choosing $|S| = \hbar/2$ to make contact with Dirac theory. The possibility of coupling this with a timelike velocity has been most thoroughly studied by Corben [10]. However, his model does not recognize the intimate connection between electron spin and phase. The alternative possibility of coupling with a lightlike velocity overcomes that deficiency. I confess to struggling with that model for many years before I realized that for a lightlike particle the spin must be a lightlike bivector. Shortly thereafter I learned that Weyssenhoff had figured that out long before [9].

There are many more things to say about the timelike case, but we shall concentrate on the more interesting lightlike case of Weyssenhoff. The tools of STA enable us to develop this case much farther than he could, indeed, to a fairly definitive and satisfactory conclusion.

IV. FREE PARTICLE MOTION

We gain crucial insight into the "kinematics of spin" from the free particle solution. The solution has the same form for both timelike and lightlike cases, though we shall see that there are significant differences in physical interpretation.

For a free particle it follows from (33), (36) and (35) that p and $J = p \wedge z + S$ are constants of motion. We seek a general solution with constant mass $m_e = p \cdot u = p \cdot \dot{z}$. Integrating this relation, we obtain

$$p \cdot (z - z_0) = m_e \tau \,, \tag{44}$$

where z_0 is the particle position at $\tau = 0$. Angular momentum conservation gives us

$$(z - z_0) \wedge p = S(\tau) - S_0 \,, \tag{45}$$

where S_0 is the initial value of the spin. Adding these equations and multiplying by $p^{-1} = p/p^2$ we get an equation for the *particle history*:

$$z = [S(\tau) - S_0]\, p^{-1} + m_e p^{-1} \tau + z_0 \,. \tag{46}$$

We can write this in the form

$$z(\tau) = r(\tau) + x(\tau) \,, \tag{47}$$

where a center of motion is defined by

$$x = \frac{m_e}{p}\,\tau + z_0 - S_0 \cdot p^{-1} \,, \tag{48}$$

and a radius vector is defined by

$$r = S(\tau) \cdot p^{-1} \,. \tag{49}$$

Differentiating, we obtain

$$\dot{r} = \dot{S} \cdot p^{-1} = (u \wedge p) \cdot p^{-1} = u - m_e p^{-1} \,. \tag{50}$$

It follows that $\dot{r} \cdot r = 0$, so r^2 is a constant of motion, and $r = r(\tau)$ must be a rotating vector. Hence (47) describes a helix in spacetime centered on the timelike straight line (48).

We get an explicit form for the temporal behavior of S and r by introducing a rotor $R = R(\tau)$ so that

$$S = R S_0 \widetilde{R} \,. \tag{51}$$

Then (49) is consistent with

$$r = R r_0 \widetilde{R} \tag{52}$$

provided

$$p \cdot \Omega = 0 \tag{53}$$

for $\Omega = 2\dot{R}\widetilde{R}$. Furthermore

$$\dot{z} = u = R u_0 \widetilde{R} \tag{54}$$

if Ω is a constant, in which case R' integrates to

$$R = e^{\frac{1}{2}\Omega\tau} \,. \tag{55}$$

From this we conclude that all aspects of the motion are determined by the scalar, vector and bivector constants of motion, m_e, p and Ω. The values of these constants are left to be determined by other considerations.

These results apply to both timelike and lightlike cases. In the timelike case, if one assumes $|S| = \hbar/2$, it is easy to prove that

$$|\Omega| = \omega_e \equiv \frac{2m_e c^2}{\hbar} \approx 1.5527 \times 10^{21} \text{rad/sec}, \tag{56}$$

which is the *zitter frequency* of Schroedinger where m_e is the electron rest mass.

Moreover, writing $p = m_e v$, we can express the proper time in the form $\tau = v \cdot z = v \cdot x$, so (55) can be expressed in the form

$$R = e^{\frac{1}{2}\mathbf{i}\,\omega_e \tau} = e^{\mathbf{i}\,p \cdot x/\hbar} \,, \tag{57}$$

which, with **i** as a unit spacelike bivector, has been shown elsewhere [1] to correspond precisely to plane wave solutions of the Dirac equation.

The same solution applies to the lightlike case, but with the additional constraint $\omega_e \lambda_e = c$, so the *electron zitter radius* has the fixed value

$$\lambda_e \equiv \frac{\hbar}{2m_e c} \approx 1.9308 \times 10^{-3}\,\text{Å}. \tag{58}$$

Regarded as a solution of the Dirac equation, the spinor (57) describes a congruence of either timelike or lightlike histories, depending on the choice of velocity observable. Our next task is to generalize these insights to include interactions.

V. LAGRANGIAN FOR A LIGHTLIKE PARTICLE WITH SPIN

In Section III we postulated equations of motion and a number of relations among velocity, spin, momentum and mass variables. To ensure that all this constitutes a complete, coherent and self-consistent dynamical system we show that it can be derived from a single Lagrangian, but with introduction of further constraints.

Conventional Lagrangians cannot be used for lightlike particles, owing to the absence of proper time as a natural parameter. Weyssenhoff [17] addressed this problem by deriving parameter invariant Lagrangians for both timelike and lightlike particles. However, he encountered problems that limited his treatment to the free particle case. Krüger [18] resolved the problem for a lightlike particle by introducing a Lagrangian defined in terms of particle path curvature (i.e. acceleration) instead of velocity. However, his model does not have all the features that we are looking for, so we employ a different approach here.

We take full advantage of STA by using it to construct a spinor-based Lagrangian. Proca [19] was the first to use Dirac spinors for describing classical particles. Barut and Zanghi [20] used the same approach to model electron zitter. Gull [21] translated their model into STA and noted that it lacked a magnetic moment.

Here we see that spinors are especially helpful for building lightlike constraints into the Lagrangian. The resulting spinor equations provide a superior dynamical model for a lightlike particle with spin, mass and zitter, which, as we see later, is intimately related to the Dirac equation. Thus, we are able to complete Weyssenhoff's program to construct dynamical equations for a lightlike particle with zitter and electromagnetic interactions.

Continuous "motion" of a particle in spacetime is represented by a curve $z = z(\varphi)$ and its derivative $z' = dz/d\varphi$, where φ is an affine parameter for which the physical interpretation is initially unspecified. The kinematic structure of this curve is described by a spinor $\psi = \psi(\varphi)$ and its derivative ψ'.

The dynamics of motion in an external electromagnetic field $F(z) = \nabla \wedge A$ with vector potential $A = A(z)$ is determined by a Lagrangian $L = L(z, \psi, P)$ of the form

$$L = \Big\langle -\hbar \psi' \gamma_+ \gamma_1 \widetilde{\psi} + P(z' - \psi \gamma_+ \widetilde{\psi})$$
$$+ qA\psi\gamma_+\widetilde{\psi} + q\lambda_e F\psi\gamma_+\gamma_1\widetilde{\psi}\Big\rangle, \tag{59}$$

with $\gamma_\pm \equiv \gamma_0 \pm \gamma_2$ and units: $c = 1, \hbar = m_c =$ fixed reference mass. The coupling constants are charge q and a length λ_e, which we anticipate identifying as the free electron zitter radius (58), so it amounts to introducing the electron rest mass m_e as a coupling constant. A vectorial Lagrange multiplier P relates the two kinds of kinematic variable.

The *method of multivector differentiation* is the simplest and most elegant way to derive equations of motion [22]. For a Lagrangian that is homogeneous of degree one in derivatives, variation with respect to a multivector variable $X = X(\varphi)$ yields the multivector Lagrange equation

$$\delta_X L \equiv \partial_X L - \partial_\varphi(\partial_{X'} L) = 0, \tag{60}$$

where $X' = \partial_\varphi X$.

Variation of the Lagrangian with respect to P obviously gives

$$z' = \psi\gamma_+\widetilde{\psi} \equiv w, \tag{61}$$

which defines a *particle velocity* w in terms of the spinor ψ.

The result of varying position vector z is the *force law*

$$p' = qF \cdot w + \nabla F \cdot M_w, \tag{62}$$

with *momentum vector* p defined by the canonical expression

$$p \equiv P - qA, \tag{63}$$

and electromagnetic moment bivector defined by

$$M_w \equiv q\lambda_e \psi\gamma_+\gamma_1\widetilde{\psi}. \tag{64}$$

Of course, the vector derivative ∇ in (62) operates only on F and not on M.

The result of varying spinor ψ in the Lagrangian is the dynamical spinor equation

$$\hbar\psi'\gamma_+\gamma_1 = -p\psi\gamma_+ + q\lambda_e F\psi\gamma_+\gamma_1. \tag{65}$$

From this one can easily derive the equation of motion

$$S_w' = w \wedge p + F \times M_w \tag{66}$$

for a *spin bivector* defined by

$$S_w \equiv \frac{\hbar}{2}\psi\gamma_+\gamma_1\widetilde{\psi}. \tag{67}$$

One can also derive the equation

$$\frac{\hbar}{2}w' = \frac{2}{\hbar}p \cdot S_w + q\lambda_e F \cdot w, \tag{68}$$

which relates particle acceleration to spin and momentum.

The Lagrangian has delivered a system of coupled equations for particle velocity, spin and momentum. Some simplifications are still needed to facilitate physical interpretation and analysis.

VI. OBSERVABLES AND DYNAMICAL STRUCTURE

Consider implications of the canonical decomposition $\psi = (\rho e^{i\beta})^{\frac{1}{2}} R$. The rotor $R = R(\varphi)$ determines a one parameter family of Lorentz rotations

$$e_\mu = e_\mu(\varphi) = R\gamma_\mu \widetilde{R} \tag{69}$$

that transforms a fixed orthonormal frame of vectors into an intrinsic *comoving frame* following the particle. The comoving frame is coupled to the particle velocity by

$$w \equiv z' = \psi\gamma_+\widetilde{\psi} = \rho u\,, \tag{70}$$

where a new rescaled velocity has been defined by

$$u \equiv R\gamma_+\widetilde{R} = e_0 + e_2. \tag{71}$$

Thus, we identify ρ as a *time scale parameter!*

The *duality factor* $e^{i\beta}$ rotates multivectors into their duals, as shown by the equation

$$e^{i\beta}\gamma_\mu = \gamma_\mu \cos\beta + i\gamma_\mu \sin\beta = \gamma_\mu e^{-i\beta}. \tag{72}$$

Note that anticommutivity of vectors with the pseudoscalar has eliminated the duality factor from (70). However, the duality factor commutes with bivectors. So for the spin bivector we find

$$\begin{aligned}
S_w &= \frac{\hbar}{2}\psi\gamma_+\gamma_1\widetilde{\psi} = \frac{\hbar}{2}\rho R\gamma_+\widetilde{R}R\gamma_1\widetilde{R}e^{i\beta} \\
&= \frac{\hbar}{2}\rho u e_1(e^{i\beta}).
\end{aligned} \tag{73}$$

This makes the relation of spin to velocity $u = e_0 + e_2$ explicit. Further simplification is possible by eliminating the duality factor.

Note that the unit pseudoscalar can be written $i = \gamma_0\gamma_1\gamma_2\gamma_3 = e_0e_1e_2e_3 = e_0e_2e_3e_1$, so $ue_0e_2 = u$ implies $iu = -ui = ue_1e_3$. Consequently, the effect of the duality factor multiplying a null vector is equivalent to a rotation, and (73) is proportional to the form

$$\begin{aligned}
e^{i\beta}ue_1 &= ue^{-i\beta}e_1 = ue^{ie_0e_2\beta}e_1 = ue^{e_3e_1\beta}e_1 \\
&= u(e_1\cos\beta + e_3\sin\beta).
\end{aligned} \tag{74}$$

In other words, the net effect of $e^{i\beta}$ in (73) is to rotate e_1 into e_3. Since the duality factor appears in every term of the spinor equation (65), it can be eliminated by absorbing it into the rotor, as shown by

$$\psi\gamma_+ = \rho^{\frac{1}{2}}Re^{i\beta/2}\gamma_+ = \rho^{\frac{1}{2}}(Re^{\gamma_1\gamma_3\beta/2})\gamma_+,$$

so that (73) reduces to

$$S_w = \frac{\hbar}{2}\psi\gamma_+\gamma_1\widetilde{\psi} = \frac{\hbar}{2}\rho u e_1 = \rho S, \tag{75}$$

where we have introduced a *rescaled spin bivector* S. Note that the spin has a dual form

$$-iS = \frac{\hbar}{2}e_3e_1ue_1 = \frac{\hbar}{2}e_3u,$$

Thus, we can characterize the *spin bivector* by the compact equations

$$S \equiv \frac{\hbar}{2}ue_1 = isu, \qquad \text{where} \qquad s \equiv \frac{\hbar}{2}e_3 \tag{76}$$

defines a *spin vector.*

A. Momentum, mass and velocity

Particle mass relates momentum to velocity. When momentum is not collinear with velocity there are two distinct concepts of mass. First, assuming that the kinetic momentum p points to the future, we can define a positive mass m_p in the usual way: $p^2 = m_p^2 > 0$. Second, projection of momentum p onto the particle velocity u defines a *dynamical mass* $m \equiv p \cdot u$. This turns out to be the most significant mass variable.

To specify the relation of momentum to velocity more completely, we note that the comoving frame $\{e_\mu\}$ is not uniquely determined by its relation to the particle velocity in (71) and spin in (76). Consequently, we are free to constrain e_0 to lie in the $p \wedge u$ plane, as specified by the equation

$$p \wedge u \wedge e_0 = p \wedge e_2 \wedge e_0 = 0. \tag{77}$$

Hence, we can introduce mass parameters m_1 and m_2 so that

$$\begin{aligned}
&p = m_1e_0 - m_2e_2, \\
&\text{with} \qquad m \equiv p \cdot u = m_1 + m_2.
\end{aligned} \tag{78}$$

The various masses are thus related by

$$\frac{m_p^2}{m^2} = \frac{p^2}{(p \cdot u)^2} = \frac{m_1^2 - m_2^2}{(m_1 + m_2)^2} = \frac{m_1 - m_2}{m_1 + m_2}. \tag{79}$$

Functional values for the masses will be determined by the equations of motion.

Now the relation of momentum to particle velocity can be expressed in the simple form

$$up = m(1 + e_2e_0) \quad \text{with} \quad u \wedge p = me_2e_0. \tag{80}$$

Likewise, it follows from (78) that $p \cdot e_1$ and $p \cdot e_3 = p \cdot s = 0$. Hence, the relation of momentum to spin is reduced to the simple equations

$$p \cdot S = \frac{\hbar}{2}p \cdot (ue_1) = \frac{\hbar}{2}me_1, \tag{81}$$

$$p \wedge S = p \wedge (isu) = ip \cdot (us) = ims$$
$$\text{or} \quad ms = -i(p \wedge S) = p \cdot (iS). \tag{82}$$

These relations considerably simplify the velocity equation of motion (68) and, as we see later, the spinor equation (65) by reducing direct coupling with momentum to the scalar variable $m = p \cdot u$. One more simplification is needed.

B. Time scaling and gauge transformations

For purposes of measurement, we must relate the affine parameter φ to some observable time scale. This can be done by projecting the electron's null velocity onto a *timelike reference history* $x = x(\tau)$ with *proper time* τ and *velocity*

$$v \equiv \dot{x} = dx/d\tau, \tag{83}$$

thus associating a proper time with the electron motion. The electron velocity (61) is thus rescaled to proper time by

$$\rho \equiv v \cdot w = \frac{d\tau}{d\varphi}, \quad \text{so} \quad w = \frac{d\tau}{d\varphi}\frac{dz}{d\tau} \equiv \rho u. \tag{84}$$

Accordingly, the rescaled particle velocity is given by

$$\frac{dz}{d\tau} \equiv \dot{z} = u = e_0 + e_2 = R\gamma_+\widetilde{R}. \tag{85}$$

This leaves us with the problem of specifying the reference history and determining the time scale factor ρ. We have considerable leeway in choosing the reference history, the only requirement being that its relation to the electron history $z(\tau)$ be well defined. Two obvious choices for reference velocity are $v = p/m_p$ or $v = e_0$, but we shall see that they have drawbacks, so it is best to leave the option open until a decision is needed.

Recall that φ was originally introduced as an arbitrary affine parameter in the Lagrangian (59). In fact, the Lagrangian time scale depends on the spinor gauge, as we now prove. Dropping the prime notation for $d/d\varphi$ from here on, and using the prime to indicate a change of variable, we note that a time scale gauge transformation of the spinor can be defined as a *boost*

$$\psi \quad \rightarrow \quad \psi' = \psi e^{\frac{1}{2}\alpha\gamma_2\gamma_0} \tag{86}$$

which induces a scale change

$$\psi\gamma_+\widetilde{\psi} \quad \rightarrow \quad \psi'\gamma_+\widetilde{\psi'} = e^\alpha\psi\gamma_+\widetilde{\psi}. \tag{87}$$

Combining this with the change of variable

$$\frac{d}{d\varphi} \quad \rightarrow \quad \frac{d}{d\varphi'} = \frac{d\varphi}{d\varphi'}\frac{d}{d\varphi} = e^\alpha\frac{d}{d\varphi}, \tag{88}$$

we have a *time scale gauge transformation* that induces a change of the Lagrangian (59) to

$$L' = \left\langle -\hbar e^\alpha \frac{d}{d\varphi}(e^{\alpha/2}\psi)\gamma_+\gamma_1\widetilde{\psi}e^{\alpha/2}\right\rangle$$
$$+ \left\langle P(e^\alpha \frac{dz}{d\varphi} - e^\alpha\psi\gamma_+\widetilde{\psi}) + qAe^\alpha\psi\gamma_+\widetilde{\psi}\right.$$
$$\left. + q\lambda_e Fe^\alpha\psi\gamma_+\gamma_1\widetilde{\psi}\right\rangle.$$

Since the first term transforms differently from the other terms, the Lagrangian is not time scale invariant. Consequently, the time parameter must have a physical significance, though that was not imposed in writing the Lagrangian.

We can fix the time scale gauge by requiring $\rho = d\tau/d\varphi = \pm 1$, so proper time becomes our intrinsic time variable. That simplifies our model considerably. The choice is most conveniently expressed by adopting the constraint

$$\psi\widetilde{\psi} = \rho e^{i\beta} = \pm 1, \tag{89}$$

which suppresses the superfluous β variable as well. There are reasons to associate the minus sign with an antiparticle (positron), but we will not explore that possibility here and will stick with the plus sign for the rest of this paper.

We could have imposed the constraint (89) at the beginning by replacing the spinor ψ with the rotor R in the Lagrangian. That was not done for two good reasons. First, it avoids messing with the constraint $R\widetilde{R} = 1$, which complicates variational derivatives of the Lagrangian. Second, the Dirac wave function also has a factor $\rho e^{i\beta}$, so we want to understand how that relates to the similar factor here. Indeed, if the Dirac equation is derivable from a superposition of particle paths, as might be the case, then the individual proper times must be replaced by a common time variable, so ρ should emerge as a common time scale factor for the ensemble of paths.

To ascertain the significance of our gauge choice, we consider a class of gauge transformations with the form

$$\psi \quad \rightarrow \quad \psi' = \psi e^{\Gamma\chi}, \tag{90}$$

where, as in (86), the generator Γ is a constant bivector constructed from the vectors γ_μ. For $\Gamma = \gamma_2\gamma_1$, this is the form for an electromagnetic gauge transformation in the Dirac equation when it is accompanied by the transformation of vector potential:

$$\frac{q}{\hbar}A \quad \rightarrow \quad \frac{q}{\hbar}A' = \frac{q}{\hbar}A + \nabla\chi. \tag{91}$$

Inserting this into the Lagrangian (59), we find gauge invariance if the following two terms cancel:

$$(\nabla\chi) \cdot (\psi'\gamma_+\widetilde{\psi'}) = w' \cdot \nabla\chi, \tag{92}$$

$$\left\langle \frac{d\psi'}{d\varphi}\gamma_+\gamma_1\widetilde{\psi'}\right\rangle = \frac{d\chi}{d\varphi}\left\langle \psi'\gamma_2\gamma_1\gamma_+\gamma_1\widetilde{\psi'}\right\rangle$$
$$= -\frac{d\chi}{d\varphi}\left\langle \psi'\widetilde{\psi'}\right\rangle. \tag{93}$$

Cancelation occurs if $\left\langle \psi' \widetilde{\psi}' \right\rangle = 1$, which is precisely our constraint (89).

This is an intriguing if not ironic result. Recall that Weyl originally introduced gauge transformations as an extension of General Relativity to incorporate length scale invariance. Subsequently, he renounced that idea and introduced the notion of gauge invariance that has become standard in quantum mechanics. Here we have come full circle to find electromagnetic gauge invariance associated with time scale invariance. Perhaps Weyl had the right idea in the first place.

C. Proper equations of motion and first integrals

Now we are prepared to reformulate our equations of motion in their simplest form, with proper time as the independent variable. Using the simplification (81), we can recast the equation (68) for particle velocity in the rescaled form

$$\dot{u} = \ddot{z} = \omega e_1 + \frac{q}{m_e} F \cdot u - \dot{\rho}u, \qquad (94)$$

where a *dynamical zitter frequency* has been defined by $\omega \equiv 2m/\hbar$.

The momentum equation (62) can be written

$$\dot{p} = qF \cdot u + \nabla\Phi, \qquad (95)$$

where the *spin potential* for the gradient force is given by several equivalent forms:

$$\Phi = \Phi(\tau, z) \equiv \frac{q}{m_e} S(\tau) \cdot F(z)$$
$$= \frac{q}{m_e} (isu) \cdot F = q\lambda_e F \cdot (ue_1). \qquad (96)$$

For future reference, we note that

$$\frac{m_e}{q} \dot{\Phi} = \frac{m_e}{q} (\partial_\tau + u \cdot \nabla)\Phi$$
$$= (\partial_\tau S) \cdot F + S \cdot (\dot{z} \cdot \nabla F) = \dot{S} \cdot F + S \cdot \dot{F}. \qquad (97)$$

Finally, from (66) we get the rescaled equation of motion for spin

$$\dot{S} = u \wedge p + \frac{q}{m_e} F \times S - \dot{\rho}S, \qquad (98)$$

wherein the coupling with momentum can be simplified by (80) to $u \wedge p = me_2e_0$.

From the three *proper equations of motion* (94), (95) and (98), we easily derive

$$\frac{d(\rho m)}{d\tau} = \frac{d(p \cdot w)}{d\tau} = \dot{p} \cdot w + p \cdot \dot{w}$$
$$= \frac{q}{m_e} \frac{d(\rho S)}{d\tau} \cdot F + \rho u \cdot \nabla\Phi = \frac{d}{d\tau}(\rho\Phi), \qquad (99)$$

where the identity (42) has been used with (97). Thus we obtain an integral of motion for the electron mass:

$$\rho m = m_e + \rho\Phi. \qquad (100)$$

The integration constant is readily identified as the free electron mass m_e, and we recognize Φ, with its several different forms (96), as a variable *mass shift* due to interaction.

To learn more about the relation between mass and momentum, we note that the common interaction term in eqs. (94) and (95) can be eliminated to get

$$\frac{d}{d\tau}(p - m_e\rho u) = \nabla\Phi - m\omega_e e_1. \qquad (101)$$

This suggests that we define a *relative momentum vector*

$$\pi \equiv p - m_e\rho u. \qquad (102)$$

It then follows that

$$\pi \cdot \dot{\pi} = \frac{d}{d\tau}\left(\frac{\pi^2}{2}\right) = \pi \cdot \nabla\Phi. \qquad (103)$$

This describes the rate that spin interaction energy is converted to mass. It vanishes when the impressed field F is constant, in which case π^2 is a constant of motion.

Since $m = p \cdot u = m_1 + m_2$, the various mass parameters are related by

$$\pi^2 = p^2 - 2m_e m\rho = m(m_1 - m_2 - 2m_e\rho). \qquad (104)$$

Thus, we have two equations of change (99) and (103) for three mass parameters $\{m_1, m_2, \rho\}$ or $\{m, m_1 - m_2, \rho\}$ By inspection, no other relations among these parameters can be derived from the dynamical equations. This leaves us with a gauge freedom to fix one relation among them. There are two possibilities with clear physical meaning. On the one hand, we could require $m_1^2 - m_2^2 = p^2 = (p \cdot u)^2 = m^2$, so there is only one mass parameter. *Alternatively, we can require $\rho = 1$. This appears to be the better choice, as it simplifies the equations of motion and, as we have already noted, it accommodates gauge invariance.* Here we have learned how it relates to mass parameters. However, before committing to a gauge choice, there is more to say about the equations of motion.

D. Rotation dynamics

So far we have established a system of three equations of motion, (94), (95) and (98), for electron velocity, momentum and spin. Now it will be shown how these three equations can be reduced to a single spinor equation. The reduction is important for two reasons: First, it simplifies solving the equations of motion. Second, it greatly facilitates comparison with Dirac theory.

Reduction is made possible by expressing the momentum in the form

$$p = m_1e_0 - m_2e_2 = Rp_0\widetilde{R},$$
$$\text{where} \quad p_0 \equiv m_1\gamma_0 - m_2\gamma_2. \qquad (105)$$

Since the velocity is given by $u = e_0 + e_2 = R\gamma_+\widetilde{R}$ and the spin is given by $S = (\hbar/2)ue_1 = (\hbar/2)R\gamma_+\gamma_1\widetilde{R}$, the

evolution of all three observables is determined by the rotation dynamics of the comoving frame $e_\mu = R\gamma_\mu \widetilde{R}$. That, in turn, is determined by a single equation of motion for the rotor $R = R(\tau)$. The normalization condition $R\widetilde{R} = 1$ implies that the rotor evolution equation has the general form

$$\dot{R} = \frac{1}{2}\Omega R, \quad (106)$$

where the bivector *rotational velocity* $\Omega = \Omega(\tau)$ is a specified function. It follows immediately that equations of motion for the e_μ are given by

$$\dot{e}_\mu = \Omega \cdot e_\mu. \quad (107)$$

Our problem is therefore to determine the functional form of Ω.

Of course, the equations of motion for velocity and spin were initially derived from the spinor equation (65). That equation is difficult to handle as it stands; however, we are now prepared to put it in more tractable form. Excluding the duality factor for reasons already explained, we can write the spinor in the form $\psi = \rho^{1/2}R$, so (105) gives us

$$p\psi = \psi p_0. \quad (108)$$

Using the identity $\gamma_0\gamma_+ = -\gamma_2\gamma_+$ we have

$$p_0\gamma_+ = (m_1 + m_2)\gamma_0\gamma_+ = m\gamma_0\gamma_+$$
$$\text{and} \quad \gamma_+ p_0 = m\gamma_+\gamma_0, \quad (109)$$

so (65) simplifies to

$$\hbar\psi'\gamma_+ = m\psi\gamma_0\gamma_+\gamma_1 + q\lambda_e F\psi\gamma_+. \quad (110)$$

The problem now is to solve for ψ'. This is complicated by the fact that one cannot divide out the null vector. The slickest way around this obstacle is as follows. First apply (109) to (110) to get

$$\hbar\psi'\gamma_+ p_0 = -m^2\psi\gamma_-\gamma_1 + q\lambda_e F\psi\gamma_+ p_0. \quad (111)$$

Then use (108) to get

$$\hbar\psi' p_0\gamma_+ = \hbar[(\psi p_0)' - \psi p_0'\gamma_+] = \hbar(p'\psi + p\psi' - \psi p_0')\gamma_+,$$

which, on inserting (110) and the momentum equation (95), becomes

$$\hbar\psi' p_0\gamma_+ = \hbar\rho(qF \cdot u + \nabla\Phi)\psi\gamma_+$$
$$+ p(p\psi\gamma_+\gamma_1 + q\lambda_e F\psi\gamma_+) - \psi p_0'\gamma_+. \quad (112)$$

Adding (110) and (112), we obtain

$$2\hbar\psi'\gamma_+ \cdot p_0 = \hbar\rho(qF \cdot u + \nabla\Phi)\psi\gamma_+$$
$$+ \psi(p^2\gamma_+ - m^2\gamma_-)\gamma_1 + q\lambda_e[2p \cdot F\psi\gamma_+$$
$$+ F\psi'(p_0\gamma_+ + \gamma_+ p_0)] - \hbar\psi p_0'\gamma_+,$$

This can be solved easily for ψ' using

$$\gamma_+ \cdot p_0 = \frac{1}{2}(\gamma_+ p_0 + p_0\gamma_+) = p \cdot u = m,$$

and further simplified with

$$p^2\gamma_+ - m^2\gamma_- = m[(m_1 - m_2)\gamma_+ - (m_1 + m_2)\gamma_-]$$
$$= mp_0\gamma_0\gamma_2,$$
$$p_0'\gamma_+ = (m_1'\gamma_0 - m_2'\gamma_2)\gamma_+ = m'\gamma_0\gamma_+. \quad (113)$$

Finally, we obtain the desired spinor equation:

$$\psi' = \frac{1}{2}\left[\frac{2}{\hbar}pe_0e_2e_1 + \frac{q}{m_e}F + \right.$$
$$\left. \frac{1}{m}\left(\rho\nabla\Phi - \frac{q}{m_e}F \cdot \pi - m'e_0\right)u\right]\psi, \quad (114)$$

where $\pi = p - m_e w = p - m_e\rho u$, as before.

Expressing (114) in the general form,

$$\psi'\psi^{-1} = \rho'/\rho + \Omega = \dot{\rho} + \Omega, \quad (115)$$

we can separately equate scalar and bivector parts. The scalar part gives us

$$\frac{\rho'}{\rho} + \frac{m'}{m} = \frac{(\rho m)'}{\rho m} = \frac{\rho}{m}u \cdot \nabla\Phi - \frac{q}{mm_e}F \cdot (p \wedge u), \quad (116)$$

which is identical to the equation (98) that gave us the mass integral of motion (99). The bivector part gives us

$$\Omega = \frac{2}{\hbar}pe_0e_2e_1 + \frac{q}{m_e}F$$
$$+ \frac{1}{m}\left(\rho\nabla\Phi - \frac{q}{m_e}F \cdot \pi\right) \wedge u + \frac{m'}{m}e_2e_0. \quad (117)$$

This is the general expression for rotational velocity that we were seeking. It holds for any mass gauge choice, including $\rho = 1$, which we now adopt for the balance of this paper.

The first term in (117) can be interpreted as a *kinetic rotational velocity*:

$$\Omega_k \equiv \frac{2}{\hbar}pe_0e_2e_1 = \frac{2}{\hbar}(m_1e_2 - m_2e_0)e_1. \quad (118)$$

It is a spacelike vector (since $\Omega_k^2 = -4p^2/\hbar^2$), so it generates a spacelike internal *zitter rotation*, as is evident from its contribution to the velocity equation: $\Omega_k \cdot u = \omega e_1$, where the zitter frequency $\omega = 2m/\hbar$ is determined by the mass equation $m = m_e + \Phi$. The second term $(q/m_e)F$ can be regarded as a generalization of the Lorentz force, as it determines extrinsic bending of the particle history due to external fields (in (94)), along with a torque on the spin (in (98)). The remaining terms account for shift in the internal rotation rate to accommodate change in mass, but they do not directly contribute to the equations for velocity and spin.

Further insight into the rotational velocity is obtained by multiplying (114) by the spin bivector (76). The $\dot{m}$ term cancels out (when $\dot{\rho} = 0$), so only the first two terms survive to give us the elegant relation

$$\Omega S = \Omega \cdot S + \Omega \times S + \Omega \wedge S = -pu + \frac{q}{m_e}FS. \quad (119)$$

Separation into parts of homogenous grade yields:

$$\Omega \cdot S = -m + \frac{q}{m_e} F \cdot S = -m_e, \qquad (120)$$

$$\Omega \times S = u \wedge p + \frac{q}{m_e} F \times S, \qquad (121)$$

$$\Omega \wedge S = \frac{q}{m_e} F \wedge S. \qquad (122)$$

Here we see that the mass integral of motion $m - \Phi = m_e = -\Omega \cdot S$ can be interpreted as constraining projection of the spin onto the rotational velocity to be a constant of motion.

It is noteworthy that (with $\rho = 1$) the mass integral of motion can be reformulated as an expression for the zitter frequency:

$$\omega = \omega_e + \frac{2}{\hbar} \Phi = \omega_e + \frac{q}{m_e} F \cdot (ue_1) = \frac{d\varphi}{d\tau}, \qquad (123)$$

where, anticipating further analysis, a *zitter phase angle* φ has been introduced. We shall see that this angle is completely analogous to the phase angle in the Dirac equation and, *á fortiori*, to the phase angle in Schroedinger's equation. Note that the phase angle may depend on the electromagnetic vector potential, but the frequency shift is gauge invariant, as it has the form of a dynamical flux integral on the circulating particle history. The present context also suggests that the phase φ may be a more fundamental time variable than the proper time.

Generalizing the free particle case (58), we identify

$$\lambda = \omega^{-1} = \frac{d\tau}{d\varphi}, \qquad (124)$$

as the variable *zitter radius*. This interpretation is supported by the form of equation (94), wherein λ can be identified as an intrinsic *radius of curvature* for a helical particle history.

The *first curvature* κ_1 of a particle history measures the bending rate orthogonal to its velocity. In the particle equation of motion (94) the direction of helical bending is given by the *zitter vector* e_1. Hence, the first curvature is given by

$$\kappa_1 \equiv -\dot{u} \cdot e_1 = \omega - \frac{q}{m_e} F \cdot (ue_1) = \omega_e = \lambda_e^{-1}. \,(125)$$

According to (101) this quantity is rigorously constant! However, it is composed of two parts: the first can be regarded as an *intrinsic curvature* while the second is an "extrinsic curvature" due to external forces. This tells us that in response to external forces the intrinsic curvature is adjusted to maintain an overall constant value. Its inverse is the curve's radius of curvature, which we recognize as the free particle zitter radius.

VII. ANALYSIS AND IMPLICATIONS

The formulation of our model for a lightlike electron with zitter is now complete. In principle, the model can be applied to any problem in quantum mechanics. The present discussion is limited to a general analysis of solution techniques and a survey of promising possibilities for applications and experimental test of the model.

A. Zitter center

An important unresolved issue is how best to define the relation between the timelike reference history $x = x(\tau)$ and the lightlike particle history $z = z(\tau)$. Relative to the instantaneous rest frame defined by $v = \dot{x}$, at any time τ the two histories are separated by a displacement vector

$$r \equiv z - x. \qquad (126)$$

If this is defined so that $|r| \approx \lambda_e$, we can regard the reference history as specifying a *center of curvature*, around which the helical particle revolves as time progresses. An essential condition on a *zitter center history* $x(\tau)$ is

$$\dot{x} \cdot u = \dot{x} \cdot \dot{z} = 1, \qquad (127)$$

as that defines the proper time. What we need is a tractable equation of motion for $\dot{x}$ that satisfies these constraints. This is akin to defining an equation of motion for the guiding center of a helical orbit in plasma physics. Two possibilities present themselves immediately: $\dot{x} = p/m_p$ or $\dot{x} = e_0$. Unfortunately, their equations of motion imply that they both "wobble" with the zitter frequency, and if we try to smooth that out (below), there is no guarantee that the history $x(\tau)$ cannot drift outside the helix $z(\tau)$.

An alternative with attractive physical interpretation is to specify $x(\tau)$ by defining a zitter radius vector $r \equiv -\lambda e_1$ with variable length specified by (124). However, since the first curvature κ_1 is a constant of the motion, a better choice may be $r_e \equiv -\lambda_e e_1$, defining a zitter radius vector of fixed length. The felicity of that choice is confirmed by noting that it gives the spin potential (96) the perspicuous form

$$\Phi = qF \cdot (u \wedge r_e). \qquad (128)$$

As there is no crucial physical issue involved, we can defer choice among the alternatives to be decided by convenience in specific applications. The best choice may well depend on what approximations we adopt.

B. Zitter averages and approximations

As zitter fluctuations are so rapid, it is most convenient to separate them from zitter means, which are more

directly observable. The velocity $v = \dot{x}$ defines an instantaneous rest frame for the electron at each time τ, so we define the zitter mean as an average over the free particle zitter period that keeps the *zitter center velocity* v and the spin vector $s = (\hbar/2)e_3$ fixed. With an overline to denote average value, basic zitter means are specified by the expressions

$$\overline{v} = v = \dot{x} = \overline{u} = e_0, \quad \overline{e}_1 = 0 = \overline{e}_2, \quad \overline{e}_3 = e_3. \quad (129)$$

Consequently, the mean of the *spin bivector* is

$$\overline{S} = \frac{\hbar}{2}\overline{ue_1} = \frac{\hbar}{2}\overline{e_2 e_1} = isv. \quad (130)$$

This approximation ignores variations in zitter radius and mass over a zitter period.

Since the electromagnetic field $F(z) = F(x + r)$ acts at the location of the particle, to get its effect on the zitter center, we expand with respect to the zitter radius vector. Accordingly, the average field at the zitter center is given by

$$\overline{F(z)} = F(x) + \overline{r} \cdot \nabla F(x) + \frac{1}{2}\overline{(r \cdot \nabla)^2}F(x) + \dots \quad (131)$$

Of course, $\overline{r} = 0$, so to first order $\overline{F} = F(x)$.

Neglecting second order effects gives us the mean mass

$$\overline{m} = m_e + \overline{\Phi}, \quad \text{with} \quad \overline{\Phi} = \frac{q}{m_e}\overline{S} \cdot F(x). \quad (132)$$

The mass shift $\overline{\Phi}$ should be recognized as a generalization of the Zeeman interaction in atomic physics.

In the present approximation the mean momentum aligns with the zitter center velocity:

$$\overline{p} = \overline{m}_1 \overline{e}_0 - \overline{m}_2 \overline{e}_2 = m_e v, \quad (133)$$

where we have used

$$\overline{m}_1 = m_e, \quad \text{so} \quad \overline{m}_2 = \overline{m} - \overline{m}_1 = \overline{\Phi}. \quad (134)$$

Consequently, the momentum equation (95) can serve as an equation of motion for the zitter center. Projecting out the effect of the mass derivative and averaging, we obtain from (95) a mean equation

$$m_e \dot{v} = q\overline{F} \cdot v + v \cdot (v \wedge \nabla)\overline{\Phi}. \quad (135)$$

This will be recognized as the classical equation for a charged particle with intrinsic spin, so it can be regarded as the *classical limit of the zitter model*. With $\overline{F} = F(x)$, the first term on the right is the classical Lorentz force, while the second is the Stern-Gerlach force.

For slowly varying electromagnetic fields there is not much difference between equations (95) and (135), save that replacement of the Lorentz force $qF \cdot v$ by $qF \cdot u$ produces a *wobble* of v about its mean value. The big difference in the equations comes with electromagnetic fields oscillating with a frequency close to the zitter frequency, for then resonance can occur, as we shall see.

Our average model is not complete without an equation of motion for the spin. We can get that by computing the mean rotational velocity from (116), with the result

$$\overline{\Omega} = e_1 e_2 \omega_e - \frac{1}{m_e}v \wedge \nabla\overline{\Phi} + \frac{q}{m_e}\overline{F}$$
$$+ e_0 \left(e_1 \frac{2\overline{\Phi}}{\hbar} - e_2 \frac{\dot{\overline{\Phi}}}{\overline{m}} \right). \quad (136)$$

The last pair of terms might be regarded as a second order effect, but if the zitter vectors e_1, e_2 are "dotted out" it surely becomes first order. The mean motion of the comoving frame is now given by

$$\dot{e}_\mu = \overline{\Omega} \cdot e_\mu, \quad (137)$$

which determines the mean spin motion as well. Let's call this the *minimal zitter model*, because it is the simplest model that captures the main features of zitterbewegung. Obviously, the model can be extended by including higher order moments in the averaging.

In the next Section we see that the minimal model corresponds closely to Dirac theory, with $v = \overline{u}$ corresponding to the direction of the Dirac current, while $\overline{S}$ matches the Dirac spin. That may be surprising, since the spin bivector here satisfies the null constraint $S^2 = 0$, and no such thing appears in standard accounts of Dirac theory. The discrepancy is resolved by considering the *spacetime split* [1]:

$$S = \frac{\hbar}{2}ue_1 = \frac{\hbar}{2}isu = Sv^2$$
$$= (S \cdot v + S \wedge v)v = (m_e r + is)v, \quad (138)$$

where the zitter radius vector and mean spin are given by

$$m_e r \equiv -\frac{\hbar}{2}e_1 = S \cdot v \quad (139)$$

and

$$\overline{S} = (S \wedge v)v = isv, \quad (140)$$

with the sign chosen so the radius vector points away from the zitter center.

This is a split of the spin S into a rapidly rotating part $m_e r v$ and a slowly precessing part $\overline{S}$. The specification of spin vector here is algebraically identical to the one in Dirac theory [1]. The rapidly rotating unit vector $\hat{r} = -e_1$ is also inherent in Dirac theory, but its physical significance has been overlooked. In Dirac theory, identification of the radius vector and the spin vector as parts of the null bivector S has not been made heretofore. The implication is that the spin observable in Dirac theory is only the zitter average $\overline{S}$ of the full spin S. Nevertheless, zitter is still present in the Dirac equation as a rotating phase factor in the wave function, and we shall see that there are indeed possibilities to detect it.

The "v-split" of the spin in (138) entails a v-split of the electromagnetic dipole moment into electric and magnetic parts:

$$\frac{q}{m_e}S = \mathbf{d}_v + i\boldsymbol{\mu}_v \quad \text{with}$$

$$\mathbf{d}_v = -qrv, \quad \boldsymbol{\mu}_v = \frac{q}{m_e}sv = \frac{q}{m_e}\overline{S}. \quad (141)$$

Note that the electric dipole has constant magnitude $\pm|\mathbf{d}_v| = q\lambda_e = q\hbar/2m_e$, so it only rotates. A corresponding v-split of the external electromagnetic field has the form $F = \mathbf{E}_v + i\mathbf{B}_v$, where electric and magnetic parts are given by

$$\mathbf{E}_v = \frac{1}{2}(F - vFv), \qquad i\mathbf{B}_v = \frac{1}{2}(F + vFv). \quad (142)$$

Of course, these are electric and magnetic fields as "seen" in the instantaneous rest frame of the zitter center, not to be confused with fields in a "lab frame."

The induced mass shift can now be expressed in the physically perspicuous form

$$\Phi = \frac{q}{m_e}S \cdot F = \mathbf{d}_v \cdot \mathbf{E}_v - \boldsymbol{\mu}_v \cdot \mathbf{B}_v. \quad (143)$$

This is a general result, holding exactly in every application of the zitter model. As $\mathbf{d}_v$ is a rotating vector, this result is consistent with the well established experimental fact that the electron does not have a static electric dipole moment. Of course, if the $\mathbf{E}_v$ field is slowly varying, the effective shift is reduced to the Zeeman interaction $\overline{\Phi} = -\boldsymbol{\mu}_v \cdot \mathbf{B}_v$.

However, the spin split described here has a drawback, namely, that the reference direction v is constantly changing with time, so it is difficult to compare spin directions at different times. We show next how that drawback can be eliminated by examining the spin split more thoroughly from a different point of view.

C. Zitter and spin in the electron rest frame

Transformation of the time dependent instantaneous rest frame specified by velocity $v = v(\tau)$ to an inertial rest frame specified by a constant vector γ_0 is a *boost*, specified by

$$v = L\gamma_0\widetilde{L} = L^2\gamma_0 \quad \text{or}$$
$$L^2 = v\gamma_0 = v_0(1 + \mathbf{v}), \quad (144)$$

where $v_0 = v \cdot \gamma_0$ is the relativistic time dilation factor relating particle proper time τ to inertial time t, that is,

$$v_0 \equiv v \cdot \gamma_0 = \frac{1}{(1 + \mathbf{v}^2)^{\frac{1}{2}}} = \frac{dt}{d\tau}, \qquad \mathbf{v} = \frac{d\mathbf{x}}{dt} \quad (145)$$

is the *relative velocity* of the particle in the inertial reference frame. Let us refer to this frame as *the electron rest frame*, though we shall generalize the definition somewhat in a later subsection

We can solve (144) to get an explicit expression for the rotor L that generates the boost [1]:

$$L = (v\gamma_0)^{\frac{1}{2}} = \frac{1 + v\gamma_0}{[2(1 + v_0)]^{\frac{1}{2}}} = \frac{1 + v_0 + v_0\mathbf{v}}{[2(1 + v_0)]^{\frac{1}{2}}}. \quad (146)$$

However, it is usually easier to work with the simpler form for L^2 in (144).

The deboost of the v-split for spin S in (138) to a split in the electron rest frame is now given by

$$S = LS_0\widetilde{L} \qquad \text{where} \qquad S_0 = -m_e\mathbf{r} + i\mathbf{s}, \quad (147)$$

with electric and magnetic moments defined by

$$\frac{q}{m_e}S_0 = -q\mathbf{r} + i\frac{q}{m_e}\mathbf{s} \equiv \mathbf{d} + i\boldsymbol{\mu}. \quad (148)$$

The deboost of the particle velocity has the form

$$uv = L(1 + \mathbf{u})\widetilde{L}. \quad (149)$$

Since spin and velocity are related by $Su = 0$, from (147) and (149) we obtain

$$S_0(1 + \mathbf{u}) = (-m_e\mathbf{r} + i\mathbf{s})(1 + \mathbf{u}) = 0. \quad (150)$$

Separating parts of homogeneous grade, we ascertain

$$\mathbf{r} \cdot \mathbf{u} = 0, \qquad \mathbf{s} \cdot \mathbf{u} = 0 \quad (151)$$

and

$$\mathbf{s} = -im_e\mathbf{r}\mathbf{u} = m_e\mathbf{r} \times \mathbf{u}. \quad (152)$$

This makes spin look like a classical orbital angular momentum. However, the velocity $\mathbf{u} = \dot{\mathbf{z}}$ is not generally the derivative of the radius vector $\mathbf{r}$, as we shall see.

It is often useful to express the relation of spin to zitter in terms of a rotating orthonormal frame:

$$\mathbf{e}_k \equiv \widetilde{L}e_kvL = U\boldsymbol{\sigma}_k\widetilde{U}, \quad (153)$$

where $\mathbf{e}_1 = \hat{\mathbf{r}}$, $\mathbf{e}_2 = \mathbf{u}$ and $\mathbf{e}_3 = \hat{\mathbf{s}}$. However, it is physically more enlightening to first study the dynamics of spin and zitter more directly.

To treat interaction with external fields in the electron rest frame, we need the deboost of an electromagnetic field:

$$F_0 \equiv \widetilde{L}FL = \mathbf{E}_0 + i\mathbf{B}_0, \quad (154)$$

where

$$F = \mathbf{E} + i\mathbf{B} = LF_0\widetilde{L} \quad (155)$$

expresses the field in terms of electric and magnetic fields defined in the lab frame.

Note that

$$< SF > = < LS_0\widetilde{L}F > = < S_0\widetilde{L}FL >$$
$$= < S_0F_0 > = < (\mathbf{d} + i\boldsymbol{\mu})(\mathbf{E}_0 + i\mathbf{B}_0) > . \quad (156)$$

Whence the spin potential ($=$ mass shift) assumes the form

$$\Phi = \frac{q}{m_e}S \cdot F = \frac{q}{m_e}S_0 \cdot F_0 = \mathbf{d} \cdot \mathbf{E}_0 - \boldsymbol{\mu} \cdot \mathbf{B}_0. \quad (157)$$

However, this is not the final form required for physical interpretation.

We still need explicit expressions for the deboosted fields $\mathbf{E}_0$ and $\mathbf{B}_0$ in terms of lab fields $\mathbf{E}$ and $\mathbf{B}$. Using (144) we have, with obvious notation,

$$F_0 = \widetilde{L}FL = F_\parallel + F_\perp L^2 \tag{158}$$

where

$$F_\parallel \mathbf{v} = \mathbf{v}F_\parallel, \quad F_\perp \mathbf{v} = -\mathbf{v}F_\perp. \tag{159}$$

Whence

$$\mathbf{E}_0 = \mathbf{E}_\parallel + v_0 \mathbf{E}_\perp + v_0 \mathbf{v} \times \mathbf{B}, \tag{160}$$

where

$$\mathbf{E}_\parallel = (\mathbf{E} \cdot \hat{\mathbf{v}})\hat{\mathbf{v}}, \quad \mathbf{E}_\perp = \mathbf{E} - \mathbf{E}_\parallel. \tag{161}$$

Similarly,

$$\mathbf{B}_0 = \mathbf{B}_\parallel + v_0 \mathbf{B}_\perp - v_0 \mathbf{v} \times \mathbf{E}. \tag{162}$$

Insertion into (157) gives explicit expressions for the interaction of the electron's electric and magnetic dipoles with external fields.

Physical interpretation of spin dynamics is facilitated by transforming the spin equation of motion (98) to the instantaneous rest frame. Thus,

$$\dot{S} = L(\dot{S}_0 + \Omega_0 \times S_0) = L(m\mathbf{e}_2 + \frac{q}{m_e}F_0 \times S_0)\widetilde{L}, \tag{163}$$

where the rotational velocity Ω_0 is determined by differentiating the boost (146), with the result [1]:

$$L\Omega_0 \widetilde{L} = \Omega_v \equiv 2\dot{L}\widetilde{L} = \frac{\dot{v} \wedge (v + \gamma_0)}{1 + v_0}$$
$$= v_0 \dot{\mathbf{v}} + \frac{\dot{v}_0 \mathbf{v}}{1 + v_0} + i\frac{v_0^2 \dot{\mathbf{v}} \times \mathbf{v}}{1 + v_0}. \tag{164}$$

As is evident in equation (163), this quantity acts like an effective electromagnetic field induced by acceleration of the rest frame. The magneticlike term at the right side of the equation is responsible for the classical *Thomas precession* of the spin.

Inserting the split (147) with (148) into (163), we have

$$\dot{S}_0 = m_e \dot{\mathbf{r}} + i\dot{\mathbf{s}}$$
$$= m\mathbf{e}_2 + (F_0 - \frac{m_e}{q}\Omega_0) \times (\mathbf{d} + i\boldsymbol{\mu}). \tag{165}$$

Now we introduce the split

$$\mathbf{a} + i\mathbf{b} \equiv F_0 - \frac{m_e}{q}\Omega_0 = \widetilde{L}(F - \frac{m_e}{q}\Omega_v)L, \tag{166}$$

noting that expressions for $\mathbf{a}$ and $\mathbf{b}$ can be obtained from (164) by the deboost (158). Using

$$(\mathbf{a} + i\mathbf{b}) \times (\mathbf{d} + i\boldsymbol{\mu})$$
$$= \mathbf{a} \wedge \mathbf{d} - \mathbf{b} \wedge \boldsymbol{\mu} + i(\mathbf{a} \wedge \boldsymbol{\mu} + \mathbf{b} \wedge \mathbf{d}) \tag{167}$$

with $\mathbf{a} \wedge \mathbf{d} = i(\mathbf{a} \times \mathbf{d})$, we split (165) into coupled equations of motion for zitter and spin:

$$m_e \dot{\mathbf{r}} = m\mathbf{e}_2 + \boldsymbol{\mu} \times \mathbf{a} + \mathbf{d} \times \mathbf{b}, \tag{168}$$

$$\dot{\mathbf{s}} = \mathbf{a} \times \mathbf{d} + \boldsymbol{\mu} \times \mathbf{b}. \tag{169}$$

These equations are helpful for analyzing the dynamical behavior of spin and zitter vectors.

The spin equation (169) is most familiar. Its last term $\boldsymbol{\mu} \times \mathbf{b}$ is recognized as the usual spin precession torque, including the Thomas precession, as already noted. The term $\mathbf{a} \times \mathbf{d}$ is something new. First note that

$$\dot{\mathbf{s}} \cdot \mathbf{s} = (\mathbf{a} \times \mathbf{d}) \cdot \mathbf{s} = (\mathbf{d} \times \mathbf{s}) \cdot \mathbf{a} = 0 = \mathbf{e}_2 \cdot \mathbf{a}. \tag{170}$$

To prove that this quantity does indeed vanish, we use (166):

$$m\mathbf{a} \cdot \mathbf{e}_2 = < La\widetilde{L}u \wedge p > = < (F - \frac{m_e}{q}\Omega_v)u \wedge p >$$
$$= \frac{m_e m}{q}[\dot{u} \cdot v + \dot{v} \cdot u] = 0. \tag{171}$$

It follows from this constraint that we can write

$$\mathbf{a} \times \mathbf{d} = \hat{\mathbf{s}} \times \mathbf{d}(\hat{\mathbf{s}} \cdot \mathbf{a}). \tag{172}$$

This torque has several interesting properties. First, it rotates $\mathbf{s}$ about the vector $\mathbf{e}_2$, which is the direction of zitter circulation in the rest frame. Second, the torque decreases in magnitude until it vanishes at $\mathbf{s} \cdot \mathbf{a} = 0$. Third, its zitter average is zero if $\mathbf{a}$ is slowly varying. These properties suggest that the term is a good candidate for a spin polarization torque.

The zitter equation (168) has similar properties, of course, with the additional term $m\mathbf{e}_2$ expressing the high frequency zitter rotation. We only note that the $\boldsymbol{\mu} \times \mathbf{a}$ torque ensures that $\mathbf{r}$ rotates around $\mathbf{e}_2$ along with $\mathbf{s}$.

D. Zitter model for a static potential

The interaction of atoms and crystals with an electron is usually modeled with a static potential. For that purpose we introduce a static vector potential defined in the inertial frame of γ_0 at each time t and position $\mathbf{x}$ by

$$qA = V\gamma_0 \quad \text{where} \quad V = V(\mathbf{x}). \tag{173}$$

This determines an electromagnetic field

$$qF = \nabla V \wedge \gamma_0 = -\gamma_0 \nabla V = -\gamma_0 \wedge \nabla V \equiv -\boldsymbol{\nabla} V. \tag{174}$$

This, in turn, determines an electromagnetic force on the particle

$$qF \cdot u = \nabla V u \cdot \gamma_0 - \gamma_0 \dot{V}, \quad \text{where} \quad \dot{V} = u \cdot \nabla V. \tag{175}$$

Inserting this into the momentum equation (95) we obtain

$$\frac{d}{d\tau}(p + V\gamma_0) = -\nabla(Vu \cdot \gamma_0 + \Phi). \tag{176}$$

Next, we introduce the space-time split [1] $p\gamma_0 = p \cdot \gamma_0 + p \wedge \gamma_0 \equiv p_0 + \mathbf{p}$ and separate spatial and temporal parts. This yields a *conserved energy*:

$$E = p_0 + V \tag{177}$$

and, after factoring out $v_0 \approx u \cdot \gamma_0$, a momentum force law

$$\frac{d\mathbf{p}}{dt} = -\boldsymbol{\nabla}(V + v_0^{-1}\Phi), \tag{178}$$

where $v\gamma_0 = v_0(1 + \mathbf{v})$ with $\mathbf{v} = d\mathbf{x}/dt$ and $\mathbf{p} = mv_0\mathbf{v}$.

We still need an explicit form for the spin potential. From (175) we get

$$\Phi = \lambda_e q F \cdot (ue_1) = u \cdot \gamma_0\, e_1 \cdot \nabla V - \gamma_0 \cdot e_1\, \dot{V}. \tag{179}$$

However, it is better to use (160) and (162) in (157) to get the form

$$\begin{aligned} \Phi &= q\lambda_e[\mathbf{e} \cdot (\mathbf{E}_\parallel + v_0\mathbf{E}_\perp) + \hat{\mathbf{s}} \cdot (v_0\mathbf{v} \times \mathbf{E})] \\ \text{with} \qquad & q\mathbf{E} = -\boldsymbol{\nabla}V, \end{aligned} \tag{180}$$

where $\mathbf{e}$ denotes the unit zitter vector. This has the advantage of clearly separating the high frequency zitter from the low frequency spin contributions, and it exhibits the zitter radius λ_e as governing the strength of the interaction.

These results can be applied to experimental search for observable effects of electron zitter. As the energy conservation law (177) is identical to the usual one, zitter will be manifested only in momentum fluctuations, which are so small and rapid that they are observable only in resonances. Consider *Mott scattering* by a Coulomb field, for example. In low and high energy ranges the cross section will not be significantly affected by the very high frequency zitter rotation, so the standard result should be obtained [23]. However, in an intermediate range where the impact parameter is on the order of a zitter wavelength, the zitter structure of the particle trajectory should have a significant effect on the scattering. How big an effect awaits calculation!

As explained below, most resonances involve zitter field theory, which is beyond the scope of the present treatment. However, a new possibility for amplifying zitter resonances in crystals has recently appeared in electron channeling experiments, to which we now turn.

E. Zitter resonance in electron channeling

When a beam of electrons is channeled along a crystal axis, each electron is subject to periodic impulses from atoms along the axis. When the energy of the beam is adjusted so the crystal period matches the period of the electron's zitter dipole, a resonant interaction may be expected to alter the distribution of transmitted electrons. Indeed, a pioneering experiment in search of such a resonance has already been performed, but without knowledge of the dipole interaction mechanism described here.

The anticipated resonance was observed at close to the de Broglie frequency, which is precisely half the zitter frequency [24]. Our purpose here is to show how this result can be explained quantitatively by the zitter model. This confluence of theory and experiment provides ample grounds for repeating the experiment with greater accuracy to confirm the results and look for details suggested by the theory.

1. Experimental specifications

The anticipated resonant energy is easily calculated from the de Broglie's (circular) frequency $\omega_B = m_ec^2/\hbar$. One of de Broglie's original insights was that the frequency of a moving electron observed in a laboratory will be $\omega_L = \omega_B/\gamma$, where $\gamma = v_0$ is the relativistic time dilation factor. The distance traversed during a clock period is $d = 2\pi c\beta/\omega_L = hp/(m_ec)^2$. For the silicon crystal used in the experiment, the interatomic distance along the $< 110 >$ direction is $d = 3.84\,\text{Å}$, which implies a resonant momentum $p = 80.874$ MeV/c.

In channeling the maximum crystal potential is a few hundred electron volts at most, so in the 80 MeV region of interest to us, the effective electron mass $M \equiv \gamma m_e = E/c^2$ is constant to an accuracy of 10^{-5}, and $\gamma = 158$.

In axial channeling electrons are trapped in orbits spiraling around a crystal axis. To a first approximation, the crystal potential can be modeled as the potential for a chain of atoms, so it has the form

$$V(\mathbf{x}) = V(r, z) = U(r)P(2\pi z/d), \tag{181}$$

where $\mathbf{x}(t) = \mathbf{r} + z\boldsymbol{\sigma}_z$ is the particle position from the first atom in the chain, with $r = |\mathbf{r}|$. The longitudinal potential $P(2\pi z/d) = P(\omega_0 t)$ is periodic with a *tunable frequency* $\omega_0 = 2\pi\dot{z}/d$ that varies with the energy E. Note that at the expected resonance $\dot{z} = d/T_L$, so $\omega_0 = 2\pi/T_L = \omega_B/\gamma$ corresponds to the de Broglie frequency.

Our problem is to calculate perturbations on the transverse component of the momentum vector, as that can remove electrons from stable orbits in the beam. The transverse component of equation (178) has the familiar form of a nonrelativistic equation:

$$M\ddot{\mathbf{r}} = -\hat{\mathbf{r}}(P\partial_r U + \gamma^{-1}\partial_r\Phi), \tag{182}$$

while the longitudinal component has the form

$$M\ddot{z} = -(U\partial_z P + \gamma^{-1}\partial_z\Phi), \tag{183}$$

where now the overdot indicates differentiation with respect to "lab time" t. In the energy range of interest, it is easy to show that oscillations in the transverse velocity can be ignored, so we regard $\dot{z}$ as a constant tunable velocity close to the speed of light as already assumed above.

2. Crystal potential

To proceed further, we need an explicit model of the crystal potential. For analytic simplicity, we approximate the potential by the first two terms in a Fourier expansion with respect to the reciprocal lattice vector. So write (181) in the more specific form

$$V = U(r)(1 + \cos \omega_0 t), \qquad (184)$$

where the coefficient of the second term is set to make the potential vanish between atoms. The first term is the potential for a uniformly charged string, which (in its simplest version) has Lindhard's form [25, 26]:

$$U(r) = -k \ln[1 + (Ca/r)^2], \qquad (185)$$

where $Ca = 0.329$Å, and (with $Z = 14$ for silicon) the coupling constant $k = Ze^2/d = 52.5$ eV is the product of electron charge with charge per unit length of the string. The constant $a = 0.885 Z^{-1/3} a_0 = (0.885)(14)^{-1/3}(0.529$Å$) = .190$Å is the Fermi-Thomas screening radius, and the constant $C^2 = 3$ is a fairly accurate fit over the range of interest.

The string potential is defined as an average over atomic potentials; thus,

$$U(r) = \int_{-\infty}^{\infty} V_{\text{atom}}[(r^2 + z^2)^{1/2}] \frac{dz}{d}, \qquad (186)$$

where the screened atomic potential is given by

$$V_{\text{atom}}(R) = -\frac{Ze^2}{R} \varphi(R/a). \qquad (187)$$

To get (185), Lindhard used the screening function

$$\varphi(R/a) = 1 - [1 + (Ca/R)^2]^{-1/2}. \qquad (188)$$

Actually, the experiment is not very sensitive to the shape of the potential, so long as it is sufficient to bind the electron to an orbit around the crystal axis. In the first approximation $V = U(r)$ the projection of the electron orbit onto a transverse plane looks like a precessing ellipse or rosette. The second term $U \cos \omega_0 t$ is ignored in most channeling calculations, as it merely produces small harmonic oscillations on the radius of the precessing ellipse. However, the periodicity of the second term is essential for resonance in the zitter perturbation of interest here.

3. Classical channeling orbits

Ignoring the zitter perturbation in (182) for the time being, we seek to ascertain the effect of the periodic factor $P(\omega_0 t) = 1 + \cos \omega_0 t$ on the orbital motion. With an obvious change of notation, we can represent the radius vector in the complex form $\mathbf{r} = r e^{\mathbf{i}\theta}$, where the imaginary

$\mathbf{i}$ is the bivector generator of rotations in the transverse plane. Then equation (182) assumes the complex form

$$M\ddot{\mathbf{r}} = -U'P e^{\mathbf{i}\theta} = -\frac{U'}{r}(1 + \cos \omega_0 t)\mathbf{r}, \qquad (189)$$

with $U' \equiv \partial_r U$.

We are interested only in radial oscillations, so we use conservation of angular momentum $L = Mr^2\dot{\theta}$ to separate out the rotational motion. With the periodic driving factor omitted, equation (189) admits the energy integral

$$E_\perp = \frac{1}{2}M\dot{r}^2 + W(r)$$

$$\text{where} \qquad W(r) = \frac{L^2}{2Mr^2} + U(r). \qquad (190)$$

Let us expand this around a circular orbit of radius r_0, and for quantitative estimates take $r_0 = 0.50$Å as a representative intermediate radius. For $\mathbf{r}_0 = r e^{\mathbf{i}\theta_0}$, equation (189) gives us

$$\dot{\theta_0}^2 = \frac{U_0'}{Mr_0} = \frac{L^2}{M^2 r_0^4}, \qquad (191)$$

where $U_0' = U'(r_0)$. In terms of $x = r - r_0$, expansion of (190) gives us

$$E_\perp = \frac{1}{2}M\dot{x}^2 + W_0 + \frac{1}{2}W_0''x^2, \qquad (192)$$

where

$$W_0 = U_0 + r_0 U_0'/2, \quad W_0' = U_0' - U_0' = 0,$$
$$W_0'' = U_0'' + 3U_0'/r_0, \qquad (193)$$

with $U_0 = -18.9$ eV,

$$U' = \partial_r U = \frac{2k}{r}\left[\frac{(Ca/r)^2}{1 + (Ca/r)^2}\right], \qquad (194)$$

with $r_0 U_0' = 31.7$ eV, and

$$U'' = -\frac{U'}{r}\left[\frac{3 + (Ca/r)^2}{1 + (Ca/r)^2}\right], \qquad (195)$$

with $r_0^2 U_0'' = -76.0$ eV.

Differentiating (192) and reinserting the periodic driving factor, we obtain the desired equation for radial oscillations:

$$\ddot{x} + \Omega_0^2(1 + \cos \omega_0 t)x = 0, \qquad (196)$$

where, for mass M at the expected resonance,

$$\Omega_0^2 \equiv \frac{W_0''}{M} = \frac{3U_0' + r_0 U_0''}{Mr_0}$$
$$= \frac{3 \times 31.7 - 76.0}{80.9 \times 10^6}\left(\frac{c^2}{r_0^2}\right), \quad (197)$$

so $\Omega_0 = 4.21 \times 10^{15}\,\text{s}^{-1}$. This should be compared with the $\dot{\theta_0} = 4.75 \times 10^{15}\,\text{s}^{-1}$ from (191) and the expected resonant frequency $\omega_0 = \omega_B/\gamma = 4.91 \times 10^{18}\,\text{s}^{-1}$. We

note that the distance traveled in one orbital revolution is $d_r = 2\pi c/\dot{\theta}_0 = 3.97 \times 10^3 \text{Å} = 0.397 \mu m$. Thus, the orbit makes 2.52 revolutions in passing through the one micron crystal, so the orbital revolutions are only weakly coupled to the high frequency radial oscillations.

Equation (196) is a special case of *Mathieu's equation*:

$$\ddot{x} + q(1 + h\cos\omega t)x = 0. \tag{198}$$

According to *Floquet's Theorem*, this equation has solutions of the general form [27, 28]

$$x(t) = e^{st} \sum_{n=-\infty}^{\infty} a_n e^{i\,n\,\omega t}. \tag{199}$$

Substitution into (198) yields a three term recursion relation that can be solved for s and ratios of the Fourier coefficients a_n. In general, the *Floquet exponent* $s = s(q, h)$ is a complex constant, so it determines whether solutions are unstable or periodic.

Since $\Omega_0 << \omega_0$ in equation (196), its Floquet exponent $s = i\Omega$ is pure imaginary, and its recursion relations give

$$\Omega^2 = \frac{3}{2}\Omega_0^2, \qquad \text{and}$$

$$\frac{a_n}{a_{n-1}} = \frac{a_{-n}}{a_{-n+1}} = \frac{\Omega_0^2}{2n^2\omega_0^2} \quad \text{for } n \geq 1. \tag{200}$$

Therefore, the solution is dominated by the first order term, with the particular form

$$x(t) = a(\cos\Omega t)\cos\omega_0 t = \frac{a}{2}(\cos\omega_+ t + \cos\omega_- t),$$

$$\omega_\pm = \omega_0 \pm \Omega. \tag{201}$$

As might have been anticipated, this describes a harmonic oscillator with high frequency ω_0 and a slowly varying amplitude with frequency Ω, which is equivalent to a sum of two oscillators with frequencies $\omega_\pm$ separated by $\omega_+ - \omega_- = 2\Omega$.

We shall see that, at the resonant frequency $\omega_0 = \omega_B/\gamma$, the frequency shift $\Omega = (0.857 \times 10^{-3})\omega_0$ is the right order of magnitude to contribute to experimental effects. Moreover, this quantity has been estimated at the particular radius $r = 0.50\text{Å}$, and it may be larger by an order of magnitude for smaller radii of experimental relevance. Accordingly, a distribution of Ω values will contribute to the experiment.

A limit on the maximum radius of a channeled electron is set by the requirement that $W(r)$ in (190) must be negative for a bound orbit. A sharper limit is set by neighboring atoms. The total crystal potential can be modeled as a sum of chain potentials isomorphic to the one for the channeling axis. For larger radii perturbation from other chains can induce transition to a neighboring chain, with the result that channeled electrons jump randomly from chain to chain. We are interested in this effect only to the extent that it affects the distribution of electrons transmitted by a single channel. However,

a quantitative estimate of such transitions will not be attempted here.

In silicon, the closest chain to a $< 110 >$ channel is at a distance of 1.36Å with atoms located at positions $z_{n+1/2} = (n + 1/2)d$ alternating with the positions $z_n = nd$ along the channel. Considering the slow precession of a channeled orbit at the resonant frequency, this chain will resonate with it for hundreds of atomic steps. In fact, the interaction might lock onto the orbit to prevent precession during resonance. This may indeed have a substantial effect on channeled electrons, but we will not investigate it further here.

4. Zitter perturbations

Now we are prepared to consider the effect of zitter perturbations on the orbit. To evaluate the zitter potential Φ, we use equation (180) with $\mathbf{E}_\parallel = (\mathbf{E} \cdot \hat{\mathbf{v}})\hat{\mathbf{v}}$ and $\hat{\mathbf{v}} \approx \boldsymbol{\sigma}_z = \boldsymbol{\nabla}z$, and we ignore the spin term (because it vanishes on averaging over spin directions). By the way, we don't need to use the spin equation of motion (98) in our calculations; we only need the fact that it implies that the unit zitter vector $\mathbf{e}$ rotates rapidly in a plane that precesses slowly with the spin $\mathbf{s}$.

Inserting the electric potential into (180), we get

$$\Phi = -\lambda_e[\gamma\mathbf{e} \cdot \hat{\mathbf{r}}U'P + \mathbf{e} \cdot \boldsymbol{\sigma}_z UP']. \tag{202}$$

Since $\hat{\mathbf{r}}$ is effectively constant over a zitter period, we can make the zitter oscillations explicit by writing $\mathbf{e} \cdot \hat{\mathbf{r}} = \cos(\omega_Z t/\gamma + \delta)$ and $\mathbf{e} \cdot \boldsymbol{\sigma}_z = \sin(\omega_Z t/\gamma + \delta')$, where ω_Z is the zitter frequency. Thus,

$$\Phi = -\lambda_e[\gamma U'P\cos(\omega_Z t/\gamma + \delta)$$
$$+ UP'\sin(\omega_Z t/\gamma + \delta')], \tag{203}$$

where the smaller second term can be neglected.

The spin potential has a twofold effect on electron motion: first, as a shift in zitter frequency according to equation (123); second, as a perturbation of the momentum in equation (182). For the frequency shift we get the explicit expression

$$\omega_Z = \omega_e - \frac{\gamma U'}{m_e c}\cos(\omega_e t/\gamma + \delta). \tag{204}$$

The modulus of the oscillating term has the estimated mean value $\gamma U'c/m_e c^2 = (158)(31.7\text{eV}/0.5\text{Å})(3 \times 10^{18}\text{Ås}^{-1}/(0.511 \times 10^6\text{eV}) = 1.96 \times 10^{16}\text{s}^{-1}$. Compared to $\omega_e = 1.55 \times 10^{21}\text{s}^{-1}$, this quantity is too small by 10^{-3} to play a role in the present experiment. However, the value of U', estimated here at a radius of $r_0 = 0.5\text{Å}$, may be regarded as lower bound, as the logarithmic potential is a very soft approximation to a realistic potential, which is much sharper at the screening radius $a = 0.190\text{Å}$ where the logarithmic potential is invalid. We shall see that close resonant encounters with atomic nuclei play a dominant role in the experiment, so the oscillating term might indeed contribute a mean frequency

shift on the order of $10^{-2}\omega_e$, which could show up in experimental data. Replacement of the soft periodic modulus $P(\omega_0 t) = (1 + \cos\omega_0 t)$ by one that is more sharply peaked at the atomic sites may strengthen this conclusion. That being said, for the rest of our analysis, it suffices to assume $\omega_Z = \omega_e$.

Inserting (203) into (182), we get an explicit expression for the zitter perturbation term:

$$-\gamma^{-1}(\boldsymbol{\nabla}\Phi)_\perp = -\gamma^{-1}\hat{\mathbf{r}}\,\partial_r\Phi$$
$$= \lambda_e\hat{\mathbf{r}}\,U''P\cos(\omega_Z t/\gamma + \delta). \quad (205)$$

5. Zitter resonance

The task remains to show that this interaction can produce a resonant amplification of the electron's orbit. Inserting it into equation (182) with a convenient choice of phase and writing $\omega = \omega_Z/\gamma$, we get the equation of motion

$$M\frac{d^2\mathbf{r}}{dt^2} = -\mathbf{r}\frac{U'}{r}(1 + \cos\omega_0 t)(1 + \frac{\lambda_e}{R}\cos\omega t), \quad (206)$$

where $R \equiv -U'/U'' = r[1 + (Ca/r)^2]/[3 + (Ca/r)^2]$ is an *effective (screened) radius*. Ignoring the amplitude modulation as determined in (201), we can reduce this to a radial equation

$$\ddot{x} + \omega_0^2(1 + \frac{\lambda_e}{R}\cos\omega t)x = 0. \quad (207)$$

Of course, we can replace ω_0 in this equation by $\omega_\pm$ to get two separate resonant peaks.

For $h = \lambda_e/R$ constant, (207) is Mathieu's equation (198), so let us solve it using $h = \lambda_e/R_0 = 1.931 \times 10^{-3}/0.208 = 9.283 \times 10^{-3}$ as an approximation. For small values of h such as this, equation (207) has a first order resonance at $\omega = 2\omega_0$. An easy way to see that is to regard (207) as an equation for a driven harmonic oscillator with driving force proportional to its amplitude. In the experiment, the "natural frequency" ω_0 was adjusted by continuously varying the initial energy of the electron until a resonance was found. As for a periodically driven harmonic oscillator, resonance occurs when the driving frequency equals the natural frequency. In this case, that means $\omega - \omega_0 = \omega_0$ or $\omega_0 = \omega/2 = \omega_Z/2\gamma$, which explains why the resonance occurs at half the zitter frequency — a surprising result until contemplation shows that it is an obvious feature of parametric resonance!

More generally, it can be shown that (207) has resonances at $2\omega_0 = n\omega$, for $n = 1, 2, \ldots$. We demonstrate that explicitly for the first order resonance by truncating the series in (199) to the form [29]

$$x = e^{st}[a\cos\frac{\omega t}{2} + b\sin\frac{\omega t}{2}]. \quad (208)$$

For a resonant solution, the Floquet exponent s must be real and positive. [Actually, to get this form from (199) we should use the Floquet coefficient $s - i\omega/2$ and incorporate its imaginary part in the series. Thus, there is some ambiguity in the definition of Floquet exponent.]

To validate the trial solution and evaluate its parameters, we insert it into the differential equation. Using trigonometric identities such as $2\cos A\cos 2A = \cos A + \cos 3A$ to reduce products to sums and dropping higher order terms, we obtain

$$s\omega b + [s^s + \omega_0^2 - (\omega/2)^2 + (h/2)\omega_0^2]a = 0,$$
$$s\omega a - [s^s + \omega_0^2 - (\omega/2)^2 - (h/2)\omega_0^2]b = 0.$$

For a near resonance solution we write $\omega = 2\omega_0 + \epsilon$ and neglect second order terms to get

$$sb - \frac{1}{2}[\epsilon - \frac{1}{2}h\omega_0]a = 0,$$
$$sa + \frac{1}{2}[\epsilon + \frac{1}{2}h\omega_0]b = 0.$$

These equations can be solved for the coefficients provided

$$s^2 = \frac{1}{4}[(\frac{1}{2}h\omega_0)^2 - \epsilon^2] > 0. \quad (209)$$

Thus, we have a resonance with width

$$\Delta\omega = 2\epsilon = h\omega_0 = 9.283 \times 10^{-3}\omega_0, \quad (210)$$

or

$$\Delta p = h(80.9\,\text{MeV/c}) = 0.751\,\text{MeV/c}. \quad (211)$$

And for the amplification factor at resonance we have

$$st = \frac{1}{4}h\omega_0 t = \frac{\pi}{2}hn = 1.46 \times 10^{-2}n, \quad (212)$$

where n is the number of atoms traversed in a resonant state. Since $\ln 2 = 0.693$, this implies that the amplitude is doubled in traversing about 50 atoms.

Now note that the value of $h = \lambda_e/R_0$ used in (210) and (212) applies to only the subclass of orbits for which $r_0 = 0.50\text{Å}$. For smaller radii the values can be much larger. In principle, to get the width of the ensemble of orbits we should replace h in (210) by its average $\overline{h}$ over the ensemble. However, the result will probably not differ much from the typical value we have chosen.

Similarly, the amplitude factor in (212) will have a distribution of values, and the doubling factor will be reached much faster for orbits with smaller radii. Presumably, random perturbations (such as thermal fluctuations of the nuclei) will limit the resonant state coherence length to some mean value $\overline{n}$. Consequently, states with smaller r_0 will be preferentially ejected from the beam.

Even more to the point, the perturbation parameter $h = \lambda_e/R$ is not constant as we tentatively supposed but increases rapidly as the electron approaches a nucleus. In resonance the value of h close to each nucleus dominates the perturbation, so its effective mean value is much smaller than the estimate for constant r_0. Evidently, resonant interaction may eject electrons with small r_0 in just a few atomic steps.

6. Experimental implications

The predicted resonance width in (211) is in fair agreement with the width in the channeling experiment data [24], considering uncertainties in the value of h and such factors as thermal vibrations that may contribute to damping. Damping can only narrow the width and destroy the resonance if it is too severe.

We need to explain how the orbital resonance is manifested in the experimental measurements. Two scintillators, SC2 and SC3, were employed to detect the transmitted electrons. The larger detector SC3, with a radius about 3 times that of SC2, served as a monitor while the smaller detector served as a counter for a central portion of the beam. The measured quantity was the ratio of SC2 to SC3 counts as momentum was varied in small steps (0.083%) over a 2% range centered at the expected resonance momentum 80.9 MeV. An 8% dip was observed at 81.1 MeV. Orbital resonances may contribute to this effect in at least two ways: first, and perhaps most important, by increasing the probability of close encounter with a nucleus that will scatter the electron out of the beam; second, by increasing the duration of eccentric orbits outside the central region. Overall, resonant interactions will be strongest on electrons confined to the central region. These observations suffice for a qualitative explanation of the observed dip. A quantitative calculation will not be attempted here.

The most problematic feature of the experiment is the $0.226\,\mathrm{MeV/c}$ difference between observed and predicted resonance energies. If estimation of the experimental error was overly pessimistic, that indicates a physical frequency shift. The most likely origin for such a shift is the frequency split in (201). The experiment was not sufficiently accurate to resolve separate peaks for the two frequencies, so the peaks would merge to broaden the measured resonance width. However, the peak for ω_+ is likely to be higher than the peak for ω_- owing to greater probability for ejection from the beam. Hence, the center of the merged distribution will be displaced to a higher frequency. If this explanation is correct, then an increase in experimental resolution will separate the two peaks, and their relative heights will measure the relative probability of ejection at the two frequencies.

Though the string approximation to the crystal potential (185) is useful for semiquantitative analysis of zitter resonance, as we have seen, it breaks down completely at radii near the screening radius $a = 0.190\text{Å}$; for then, as Lindhard [26] has shown, the electron's collision time with a nucleus is comparable to the travel time between atoms. In picturesque terms, the electron begins to "feel" individual atoms rather than a continuous string. Within this domain, our analysis of electron motion remains qualitatively the same, but a more realistic crystal potential is needed for accurate quantitative estimates. A hard lower limit on the radius is determined by the mean radius of nuclear zeropoint vibrations, which Debye theory estimates as 0.05Å [25].

If the idea of zitter resonance is taken seriously, there are many opportunities for new theoretical and experimental investigations. Increasing the resolution by three orders of magnitude will open the door to refined studies of frequency shifts, line splitting, spin effects and Zeeman splitting, all of which are inherent in zitter theory [30]. As has been noted, the most straightforward prediction of the zitter model is a second order resonance near 161.7MeV/c. In a first approximation, it can be analyzed in much the same way as here, though removal of electrons from the center of the beam may be enhanced by such processes as pair creation.

A more detailed analysis of zitter resonance in channeling requires close attention to experimental conditions, so that will be addressed elsewhere. Classical particle models have long been used for channeling calculations with considerable success. Besides being simpler and more transparent than quantum mechanical models, they even often give better results at high energies. For present purposes, the zitter model differs from the usual classical model only by the zitter dipole interaction. As we see in the next Section, though zitter is inherent in the Dirac electron theory, it is unlikely that the channeling resonance effect can be derived from the Dirac equation without some modification such as projection into a Majorana state.

F. Solution by separation of variables

An obvious strategy for solving the rotor equation of motion is to separate internal zitter oscillations from the effect of external field on overall motion of the comoving frame. Spinor equations admit a natural way to do that. For a given electromagnetic field F, the rotor $R = R(\tau)$ can be factored into

$$R = LU, \tag{213}$$

where rotor L satisfies the equation

$$\dot{L} = \frac{q}{2m_e} FL. \tag{214}$$

Moreover, instead of using zitter averages as before, we can define a mean velocity $\overline{v}$ for the zitter center by requiring

$$\overline{v} = R\gamma_0\widetilde{R} = L\gamma_0\widetilde{L} \qquad \text{so} \qquad U\gamma_0\widetilde{U} = \gamma_0. \tag{215}$$

This generalizes (144) by dropping the requirement that L be a boost. However, the boost condition placed no restrictions on equations for the velocity, whereas here we have

$$m_e\dot{\overline{v}} = qF \cdot \overline{v}, \tag{216}$$

for which the relation to the momentum equation (95), or even its zitter mean (135), is problematic in general. This drawback aside, the approach has sufficient advantages to consider its use for approximate solutions at least.

Its first big advantage is evident in the transformed spin equation of motion (163), for which the Ω_0 has been adjusted to cancel F_0, so it reduces to

$$\dot{S}_0 = \frac{\hbar}{2}\frac{d}{d\tau}(\mathbf{e}_1 + i\mathbf{e}_3) = m\mathbf{e}_2. \tag{217}$$

This is equivalent to the rotor equation

$$\dot{U} = \frac{1}{2}\omega i\,\mathbf{e}_3\,U = \frac{1}{2}Ui\boldsymbol{\sigma}_3\omega$$
$$\text{with} \qquad \mathbf{e}_k = U\boldsymbol{\sigma}_k\widetilde{U}. \tag{218}$$

It has the obvious solution $U = U_0 \exp i\boldsymbol{\sigma}_3\varphi$, though $\omega = \dot{\varphi}$ must be obtained from other considerations.

A second big advantage is that exact solutions of (214) have already been found for several kinds of electromagnetic field [1, 31]. Unfortunately, they all presume that motion is along a timelike curve $x(\tau)$, whereas, in the present case $F = F(z) = F(x+r)$ is given along the lightlike curve $z(\tau)$, so problems arise in relating the curves $x(\tau)$ and $z(\tau)$.

Another difficulty is that the constraint equation relating zitter phase φ to proper time is nonlinearly related to the rotor equation of motion, as is obvious when expressed in the form

$$\dot{\varphi} = \omega_e + \frac{q}{m_e} < FLS_0\widetilde{L} > . \tag{219}$$

Consequently, a closed solution of the coupled equations is not to be expected in most cases. However, the interaction term in (219) is invariably much smaller than the constant term ω_e, so it can often be neglected or, in any case, incorporated by iteration with the equation of motion.

The complications just described disappear when F is uniformly constant, as in the case of any combination of homogeneous electric and magnetic fields. Without working out the details, let's note some special features of this case that can contribute to an elegant solution.

For constant F, equation (214) has the simple solution

$$L = \exp\left(\frac{q}{2m_e}F\tau\right). \tag{220}$$

Since $FL = LF$ in this case, (219) reduces to

$$\dot{\varphi} = \omega_e + \frac{q}{m_e} < FS_0 >= \omega_e + \frac{q}{m_e}(\mathbf{E}\cdot\mathbf{e}_1 - \mathbf{B}\cdot\mathbf{e}_3). \tag{221}$$

The initial phase can be chosen to make the angle dependence of the zitter oscillations explicit:

$$\dot{\varphi} = \omega_e - \frac{q}{m_e}\mathbf{B}\cdot\mathbf{e}_3 + \frac{q}{m_e}|\mathbf{E}\times\mathbf{e}_3|\sin\varphi. \tag{222}$$

This equation is akin to Kepler's equation in celestial mechanics relating incommensurate time variables. Since the oscillating term is so small, solution by perturbation expansion is quite satisfactory.

Furthermore, from the momentum equation of motion (95) we find immediately that $p - qF\cdot z$ is a constant of motion. And the equation of motion (101) for $\pi = p - m_e u$ reduces to

$$\dot{\pi} = \Omega\cdot\pi = -m\omega_e e_1, \tag{223}$$

so $\pi^2 = p^2 - 2m_e m$ is another constant of motion.

Finally, a nice trick for integration to get an explicit equation for the particle history is given in [31]. This is all we need for the complete solution of motion in arbitrary homogeneous fields. Note that complications of solving for the wobbling momentum vector have been avoided.

VIII.   ZITTER RESONANCE AND FIELD THEORY

Like the original Dirac theory, our zitter model describes a test particle, which is to say that the electromagnetic field that it generates is ignored. But if we take the zitter charge circulation seriously, it is evident that the particle must be the source of a fluctuating electromagnetic field. The average of that field over a zitter period will give us the static field of a magnetic dipole and an effective Coulomb field with a virtual source at the zitter center. Superimposed on these mean values we have an electric dipole field rapidly rotating with the zitter frequency. This picture raises many questions that cannot be answered without constructing a complete electromagnetic field theory with zitter sources. That is too much to attempt here, but we can identify some of the issues and prospective physical implications.

We consider here only a classical field theory, leaving issues of second quantization and many particle theory for another day. The first question is: What maintains the zitter charge circulation? Classical electrodynamics implies that all accelerating charges radiate, which lead to collapse of circulation. An obvious suggestion is that circulation is maintained by self interaction of the charge with its own field! Of course, that cannot be regarded as a satisfactory answer until the self-interaction is calculated, and no such calculation has ever been successful in any version of classical theory. Nevertheless, it does provide a new constraint on the self-interaction problem, namely, that its solution should account for electron spin as well as mass. Besides that, there are tricky questions about singularities in the field of a lightlike source. Evidently some new idea will be needed to make zitter field theory work. However, we can take an intermediate stance, presuming the zitter field as real and exploring its implications.

The dipole interaction $\mathbf{d}\cdot\mathbf{E}$ is, of course, unobservable for laboratory electric fields, because $\mathbf{d}$ rotates so rapidly. However, if $\mathbf{E}$ is the oscillating zitter field of an electron, the possibility of resonance exists. The following *speculative survey* suggests that many surprising features of quantum mechanics might be consequences of zitter resonances.

Electron diffraction *as resonant momentum transfer mediated by the zitter field.* An electron incident on

a crystal is preceded by its zitter field which is reflected back from a periodic array of scattering centers. At the Bragg incidence angles, phase coherence of the reflected field oscillations with the electron zitter provides a mechanism for resonant momentum transfer. The propagating zitter field has a wavelength that may be identified with the electron *de Broglie wavelength* in quantum mechanics.

Quantized atomic states might be explained by the same mechanism as diffraction, that is, as resonances of the electron zitter with its own field reflected off the nucleus. This raises questions about the topology of the zitter field that may be related to *Berry phase* and the *Aharonov-Bohm effect.* Each electron in a multi-electron atom will have its own zitter frequency, shifted by its interaction energy in the atom. Consequently, an atom or molecule will have a zitter field composed of the multiple frequencies of its constituents, and it will have not a single de Broglie wavelength but a spectrum of wavelengths. Perhaps this spectrum will show up in diffraction experiments.

The **Pauli Principle** *as resonant phase locking.* In general, the zitter oscillations of different electrons will not be coherent, because of doppler shifts and frequency shifts due to local interactions. However, when two electrons are in the same quantum state the necessary conditions for resonant phase locking are fulfilled. It remains to be proved that antiparallel alignment of spins is necessary for a stable state.

Van der Waals and Casimir forces are generally explained as effects of coherent coupling between fluctuating dipoles. This involves coherent coupling of phases, which is what zitter is all about. The zitter field serves as a mechanism for maintaining that coupling that may have additional consequences. As the dipoles are mutually induced, the interaction mechanism appears to be essentially the same as in electron diffraction.

The **Lamb Shift** is commonly attributed to a smearing out of electron position due to vacuum fluctuations. Obviously, the zitter offers an alternative explanation for smearing out. Indeed, one can conceive an electromagnetic vacuum field composed of a stochastic combination of zitter fields of all existing charged fermions.

Similar speculative explanations can be adduced for tunneling, anomalous magnetic moment, covalent bonding and other important quantum effects. Even Bose-Einstein condensates might be explained by resonant zitter coupling of constituent fermions. Zitter field theory must provide reasonable accounts of all these phenomena before it can be regarded as a viable extension of standard quantum mechanics. However, we don't have to wait for that! If zitter is a real physical phenomenon we should be able to devise experiments to observe it directly. In fact, as we have noted, that may already have been done!

IX. ZITTERBEWEGUNG IN DIRAC THEORY

This section describes the intimate relation of the zitter model to the Dirac equation. The relation should not be surprising, as it was study of Dirac theory that led to the zitter model in the first place. Conversely, we shall see that the zitter model suggests modification of Dirac theory to incorporate deeper zitter substructure, with new physical implications including a surprising possible connection to weak interactions.

Our first task is to match up variables and dynamical equations in the zitter model with observables and dynamics in Dirac theory. An exact match is not to be expected, as the particle based zitter model can only be related to the Dirac field theory by some sort of averaging or projection process. Nevertheless, the comparison reveals specific similarities and differences that must be addressed in establishing a firm connection between zitter model and Dirac theory. This leads to suggestions for modifying the Dirac theory and further research to resolve outstanding issues.

In the language of STA, the *real Dirac equation* has the form

$$\gamma^\mu \partial_\mu (\psi \gamma_2 \gamma_1 \hbar - q A_\mu \psi) = m_o \psi \gamma_0,$$
$$\text{or} \quad \nabla \psi i \boldsymbol{\sigma}_3 \hbar - q A \psi = m_e \psi \gamma_0, \qquad (224)$$

where the Dirac wave function is a *real spinor field*

$$\psi = \psi(x) = (\rho e^{i\beta})^{1/2} R. \qquad (225)$$

This version of the Dirac equation is fully equivalent to the standard matrix version [1], but it has great advantages for analyzing the structure of the Dirac theory as shown in the following.

The Dirac wave function determines a frame field of *local observables*

$$\psi \gamma_\mu \tilde{\psi} = \rho e_\mu, \quad \text{where} \quad e_\mu = R \gamma_\mu \tilde{R} = e_\mu(x) \qquad (226)$$

and $\rho = \rho(x)$ is interpreted as a scalar probability density, in accordance with the interpretation of the *Dirac current* $\psi \gamma_0 \tilde{\psi} = \rho e_0$ as a probability current. The vector fields

$$v \equiv e_0 = R \gamma_0 \widetilde{R} = v(x), \qquad (227)$$

and

$$s \equiv \frac{1}{2} e_3 = \tfrac{1}{2} \hbar R \gamma_3 \widetilde{R} = s(x) \qquad (228)$$

are interpreted as local velocity and spin observables for the electron. Note that these quantities are algebraically identical to the expressions for velocity and spin in the zitter model. Likewise for the other observables

$$u \equiv e_0 + e_2 = R \gamma_+ \widetilde{R} = u(x), \qquad (229)$$

and

$$\overline{S} \equiv isv = \tfrac{1}{2} R i \boldsymbol{\sigma}_3 \hbar \widetilde{R} = \overline{S}(x). \qquad (230)$$

The difference is that the observables here are vector and bivector fields, whereas in the zitter model they are defined on a particle history. Our next task is to compare them dynamically.

A. Zitterbewegung along Dirac histories

The mass term in the Dirac equation (224) can be written in the form

$$m_e \psi \gamma_0 = m_e (\rho e^{i\beta})^{1/2} R \gamma_0 \widetilde{R} R = m_e v e^{-i\beta} \psi.$$

Whence, the Dirac equation can be reformulated in the compact form

$$D\psi \equiv \gamma^\mu D_\mu \psi = 0, \tag{231}$$

where a gauge invariant *coderivative* is defined by

$$D_\mu = \partial_\mu + \tfrac{1}{2}\Omega_\mu, \tag{232}$$

with the bivector-valued connexion

$$\Omega_\mu \equiv \frac{2}{\hbar}(m_e v_\mu e^{-i\beta} + q A_\mu) e_2 e_1. \tag{233}$$

The purpose of introducing this coderivative is not to reduce the Dirac equation to the maximally compact form (231), but to reveal that all the essential physics is contained in the connexion Ω_μ.

Our aim is to compare zitter dynamics along the history of a zitter center with Dirac dynamics along "streamlines" of the Dirac current. To that end, we evaluate the *directional coderivative* along a Dirac history as follows. From

$$D\psi\gamma_0 = D(v e^{-i\beta}\psi) = [D(v\psi) - i(\nabla\beta)v\psi]e^{-i\beta} = 0$$

we obtain

$$D(v\psi) = 2v \cdot D\psi + (Dv)\psi = i(\nabla\beta)v\psi. \tag{234}$$

This separates into the familiar conservation law for the Dirac current and a dynamical equation for rotations along a Dirac streamline:

$$\dot{\rho} + \rho D \cdot v = D \cdot (\rho v) = \nabla \cdot (\rho v) = 0, \tag{235}$$

$$v \cdot DR = \dot{R} + \tfrac{1}{2}\Omega(v)R = -\tfrac{1}{2}[D \wedge v + i(v \wedge \nabla\beta)]R. \tag{236}$$

The overdot indicates the directional derivative $v \cdot \nabla$ and (233) gives us

$$\Omega(v) = (\omega_0 \cos\beta + \frac{2q}{\hbar}A \cdot v)e_2 e_1 + (\omega_0 \sin\beta)e_0 e_3 \tag{237}$$

$$D \wedge v = \nabla \wedge v + (\omega_0 \sin\beta)e_3 e_0. \tag{238}$$

When these are inserted into (236), the boosts in the $e_0 e_3$ plane cancel to give us

$$\dot{R} = \tfrac{1}{2}[(\omega_0 \cos\beta + \frac{2q}{\hbar}A \cdot v)e_1 e_2 - \nabla \wedge v - i(v \wedge \nabla\beta)]R. \tag{239}$$

This is an exact result. It does indeed exhibit the familiar zitter rotation in the $e_2 e_1$-plane, though the frequency seems to be different.

We still need to evaluate the curl of the velocity field to appreciate its effect on the dynamics. A general expression has been derived elsewhere [2, 3], but our purpose here is served by the eikonal approximation, expressed by

$$\nabla\psi\gamma_2\gamma_1\hbar = \nabla(\psi_0 e^{-\gamma_2\gamma_1\Phi/\hbar}\gamma_2\gamma_1\hbar) = (\nabla\Phi)\psi. \tag{240}$$

Inserting this into the Dirac equation (224), we obtain

$$\nabla\Phi = m_e v e^{-i\beta} - qA. \tag{241}$$

This implies $e^{i\beta} = \pm 1$, where the choice of sign depends on the chosen sign of charge. We adopt that approximation only in this equation, as the parameter β is too important to ignore completely. Then the curl of (241) gives us

$$-m_e \nabla \wedge v = q\nabla \wedge A = qF. \tag{242}$$

This is equivalent to the Lorentz force equation for a fluid of charge with uniform density, as seen by "dotting" with v, to get

$$v \cdot (\nabla \wedge v) = v \cdot \nabla v = -\frac{q}{m_e}v \cdot F = \frac{q}{m_e}F \cdot v. \tag{243}$$

More generally, we insert (242) into (236) and use (238) to get the spinor equation of motion

$$\dot{R} = \tfrac{1}{2}[\omega_v e_1 e_2 + \frac{q}{m_e}F - i(v \wedge \nabla\beta)]R, \tag{244}$$

with $\omega_v \equiv \omega_0 \cos\beta + (2q/\hbar)A \cdot v$. This equation must be compared with the analogous expression (138) for rotational velocity in the minimal model. The general form is very similar, but we are not equipped to account for the differences for reasons to be discussed.

The zitter frequencies in the two equations are not equivalent, but they do have the same free particle limit ω_0. The bivector $i(v \wedge \nabla\beta)$ also contributes to the rotation rate, as implied by

$$v \cdot [i(v \wedge \nabla\beta)] = -i(v \wedge v \wedge \nabla\beta) = 0.$$

However, the physical significance of this term remains obscure. The apparent absence of a Stern-Gerlach force in the Dirac version (244) is noteworthy, but we cannot be sure that it is not buried in terms that we do not understand.

It might be thought that the Dirac equation is more fundamental than the zitter model because interaction comes from the vector potential A alone and interaction with the field F arises only indirectly, for example in the manner described above. A famous consequence of this is the derivation of the gyromagnetic ratio $g = 2$. In contrast, the Lagrangian (59) for the zitter model appears to presume the electron magnetic moment with separate coupling constants for A and F interactions. Note, however, that precisely two independent constants are presumed in both models. The rest mass is presumed in the Dirac equation, but that is replaced by the zitter radius in the zitter model. It remains to be seen which is more fundamental.

B. Zitterbewegung substructure

In preceding sections we saw that a rotating electric dipole is the hallmark of zitterbewegung, so one wonders why it has attracted so little attention in accounts of Dirac theory. In the original paper introducing his equation [32], Dirac concluded that the electron has both a magnetic and an electric moment, the magnetic moment being the same as in the Pauli model. However, he said, "The electric moment, being a pure imaginary, we should not expect to appear in the model. It is doubtful whether the electric moment has any physical meaning."

It is worth translating Dirac's argument into STA, especially since its mathematical content has been retained in the current literature. One simply "squares" the operator on the wave function in (224) to get

$$(-\hbar^2\nabla^2 + q^2 A^2)\psi - 2q(A\cdot\nabla + \tfrac{1}{2}F)\psi i\boldsymbol{\sigma}_3\hbar = m_e^2\psi\,,\quad(245)$$

where $F = \nabla A = \nabla \wedge A$ with the *Lorenz condition* $\nabla \cdot A = 0$. This is the Klein-Gordon equation with an extra term that explicitly shows the action of bivector F "rotating" the wave function. The interaction energy density associated with this term is proportional to

$$\tfrac{1}{2} < F\psi i\boldsymbol{\sigma}_3\hbar\widetilde{\psi} >\; = \; < F\overline{S}\rho e^{i\beta} >$$
$$= -\rho(\mathbf{B}_v\cdot\mathbf{s}\cos\beta + \mathbf{E}_v\cdot\mathbf{s}\sin\beta), \quad(246)$$

where the v-split introduced in (142) has been used on the right hand side. The $\mathbf{B}_v\cdot\mathbf{s}\cos\beta$ term is recognized as the Pauli term except for the strange $\cos\beta$ factor. The $\mathbf{E}_v \cdot \mathbf{s}\sin\beta$ term is what Dirac identified as an imaginary dipole moment. He never mentioned the electric dipole again. In his influential textbook [33] he simply suppressed the offending term by a subterfuge advertised as a change in representation. Then he killed the term with an approximation that amounts to $\sin\beta = 0$ and never looked back. We find an alternative resolution of this "dipole problem" below.

Dirac was soon convinced by Schroedinger [5] that zitterbewegung is foundational to electron theory and he argued the case vigorously in his textbook [33]. As his argument is still widely accepted [34], it deserves comment here.

Dirac introduces a position operator by identifying a velocity operator as its time derivative, and he followed Schroedinger in integrating the equation for the free particle case. He identifies his α_k matrices as velocity operators and claims that their eigenvalues ± 1 correspond to measured values of electron velocity, asserting: "we can conclude that *a measurement of a component of the velocity of a free electron is certain to lead to the result* $\pm c$." (Dirac's italics) From the STA point of view this argument and its implications are bogus, for reasons explained elsewhere [1]. However, we agree with the assumption that the electron moves with the speed of light. We differ in identifying the local electron velocity with the null vector $u = R\gamma_+\widetilde{R}$ introduced above.

We also agree with Dirac in attributing the origin of spin to zitter. Dirac concludes: " Our argument is valid only provided the position of the particle is an observable. If this assumption holds, the particle must have a spin angular momentum of half a quantum." However, Dirac's analysis of zitter and spin never went beyond the free particle case. He overlooked (or dismissed) the inference that his account of spin arising from charged particle circulation implies a rapidly rotating electric dipole. Though zitter obviously arose from wave function phase oscillations in Schroedinger's free particle analysis, Dirac never considered a general connection of zitter circulation to wave function phase. The present study can be regarded as an extension of Dirac's analysis to incorporate these features in a general theory of zitter in quantum mechanics.

As the Dirac equation has an unsurpassed record of success in QM and QED applications, it is imperative to reconcile it with any proposals about zitter. If the zitter model describes substructure in electron motion that is not captured by the Dirac equation, it must at least be related to the Dirac wave function by some sort of averaging process. Without attempting a definitive reconciliation, let us note some issues that must be addressed.

We know that the conserved Dirac current ρv determines a congruence of curves (or streamlines) for every solution of the Dirac equation. As Bohm and Hiley have argued at length [4], each of these curves $x = x(\tau)$ can be regarded as a possible path for the electron weighted by a relative probability $\rho = \rho(x)$ that the electron actually followed that path. This is a viable particle interpretation of quantum mechanics. However, a refinement is necessary to account for zitterbewegung, which suggests that the actual particle paths are lightlike helices with tangent vector $u = u(x)$ at each spacetime point. The simplest refinement would have each of these lightlike paths winding around a Dirac streamline, but this possibility is questionable without deriving it from the zitter model, at least approximately, by a well defined averaging process (yet to be determined)!

A crucial problem is to justify the weighting of paths by the probability density $\rho = \rho(x)$. A new possibility is suggested by the fact that the analogous quantity in our zitter particle model is a timescale factor $\rho = \rho(\varphi) = \frac{d\tau}{d\varphi}$, which, as we have noted, would arise naturally from averaging over paths with different proper times but a common phase angle φ. Thus, the putative probability density in the Dirac equation might be derivable as a time scale weighting on a congruence of particle paths! In any case, time scaling in the zitter model must be reconciled with the probability interpretation in the Dirac theory.

It seems likely that a suitable averaging process relating particle histories to the Dirac equation will involve time averaging as well as ensemble averaging with constraints. Analogy with the zitter model already suggests that Dirac observables for velocity v and spin $\overline{S}$ correspond to zitter time averages, but the phase of the Dirac wave function is directly comparable to the zitter phase.

The Schroedinger equation is a nonrelativistic approximation to the Dirac equation that freezes spin but preserves zitter oscillations in the phase of the wave function and in coupling of phase to amplitude, as specified, surprisingly, by the mysterious parameter β [31, 35]. As shown in these references, assuming $\beta = 0$ completely decouples phase from amplitude and so eliminates all QM effects. Clearly, therefore, the role of β must be accounted for in any averaging process.

The fact that the kinematic state of a particle with zitter is described by a rotor while the Dirac wavefunction is also a spinor suggests that the QM *superposition principle* can be construed as an average over particle rotor states. The undeniable success of the Schroedinger equation suggests that, approximately at least, the superposition is an average over phase factors, such as Feynman's sum over paths. This certainly produces the coupling of phase to amplitude so characteristic of QM. Note that an average over rotors blurs any average over paths, because path velocity is a bilinear function of the path rotor. Finally, it should be mentioned that gauge invariance provides a strong constraint on assembly of phases for distinct particle paths into a coherent ensemble.

Clearly, deriving the Dirac equation from zitter substructure is a nontrivial problem. Happily, we don't have to wait for a solution to make progress in studying the zitterbewegung. Here is a promising alternative approach:

C.   Putting zitterbewegung into the Dirac equation

We have seen that physical interpretation of the Dirac equation is crucially dependent on identification of a particle velocity observable, which requires theoretical assumptions beyond the Dirac equation itself. Historically, the mass term in the Dirac equation led to the conservation law for the Dirac current and its interpretation as a probability current, with an implicit identification of particle velocity. In the STA version of the Dirac equation (224), explicit appearance of the vector γ_0 shows that the velocity vector $v = R\gamma_0\widetilde{R}$ is inherent in the structure of the equation. However, we have identified the vector $u = R\gamma_+\widetilde{R}$ as a better candidate for electron velocity. This suggests a slight modification of the Dirac equation to replace v by u. Accordingly, we change the mass term $m_e\psi\gamma_0$ to

$$m_e\psi\tfrac{1}{2}(\gamma_0 + \gamma_2) = m_e\psi_+\gamma_0, \qquad (247)$$

where

$$\psi_+ \equiv \psi\tfrac{1}{2}(1 + \boldsymbol{\sigma}_2). \qquad (248)$$

In common parlance, this is a projection of a 4-component spinor ψ into a 2-component spinor ψ_+. As only these components are now relevant to the electron velocity, we should perform the same projection on the other components of the Dirac equation. That requires modification of the vector potential term to achieve a 2-component equation. Thus we arrive at the modified Dirac equation

$$\nabla\psi_+i\boldsymbol{\sigma}_3\hbar - qA\psi_+\boldsymbol{\sigma}_3 = m_e\psi_+\gamma_0. \qquad (249)$$

Let's refer to this as the *zitter Dirac equation*. Note that it is invariant under projection from the right by $\frac{1}{2}(1 - \boldsymbol{\sigma}_2)$.

To be assured that this modification of the Dirac equation has not damaged its essential physical meaning, we note that the zitter free particle solution (57) is also a solution of the Dirac equation, which, when substituted into (249) with arbitrary initial conditions and $\mathbf{i} = i\boldsymbol{\sigma}_3$, yields the algebraic relation

$$pu = m_e(1 - e_2e_0). \qquad (250)$$

This is identical to the relation between momentum and velocity found in (80) for the zitter model. In contrast to the prosaic relation $p = m_e e_0$ from the free particle solution to the ordinary Dirac equation, this relation includes zitter in the vector e_2, as it rotates with the zitter phase. Thus, the form of the zitter Dirac equation brings the null velocity observable to the fore.

It is also readily shown that the zitter Dirac equation has the same electromagnetic gauge invariance as the ordinary Dirac equation, though we shall see that the gauge generator assumes a different form when the gauge group is generalized to incorporate electroweak interactions.

Now check the observables. With respect to the projected wave function, the observables (229), (230) become

$$\psi_+\gamma_+\widetilde{\psi}_+ = 2\psi_+\gamma_0\widetilde{\psi}_+ = \psi\gamma_+\widetilde{\psi} = \rho u, \qquad (251)$$

and

$$\begin{aligned} \psi_+i\boldsymbol{\sigma}_3\hbar\widetilde{\psi}_+ &= \psi\tfrac{1}{2}(1 + \boldsymbol{\sigma}_2)i\boldsymbol{\sigma}_3\hbar\psi \\ &= \frac{\hbar}{2}\psi\gamma_+\gamma_1\psi = \rho\frac{\hbar}{2}ue^{-i\beta}e_1 = \rho S. \end{aligned} \qquad (252)$$

Thus we get the same zitter velocity u, but the spin bivector $\overline{S}$ is replaced by a null spin bivector S, which is identical in form to the spin (73) in the point particle model, and, as in that case, the duality factor can be absorbed into a rotation of the vector e_1. We see immediately that the interaction energy density (246) becomes

$$\begin{aligned} \tfrac{1}{2} < F\psi_+i\boldsymbol{\sigma}_3\hbar\widetilde{\psi}_+ >&=< F\rho S > \\ &= \rho(\mathbf{E}_v \cdot \mathbf{d} - \mathbf{B}_v \cdot \mathbf{s}), \end{aligned} \qquad (253)$$

in perfect accord with equation (143) of the zitter model. Thus, the change in observables by projection on the wave function appears to eliminate the parameter β and its problems of physical interpretation.

Further insight comes from the following Lagrangian for the zitter Dirac equation:

$$\begin{aligned} L_{zD} = 2\Big\langle[-\hbar\nabla\psi_+i\gamma_0 \\ +qA\psi_+\gamma_0 + m_e\psi_+\boldsymbol{\sigma}_3]\widetilde{\psi}\Big\rangle. \end{aligned} \qquad (254)$$

Note that the interaction term has the usual form $A \cdot J = \langle AJ \rangle$, but now the charge current is a null vector field

$$J = 2q\psi_+\gamma_0\widetilde{\psi} = q\psi\gamma_+\widetilde{\psi} = q\rho u, \qquad (255)$$

as expected. It follows from the zitter Dirac equation that this current is conserved, though it is not conserved in standard Dirac theory.

The usual Dirac current is not obtainable as a bilinear observable of the wave function ψ_+. However, as noted before, it can be obtained as a zitter average $v = \overline{u}$ of the zitter velocity. This suggests that one should try to derive the zitter Dirac equation, rather than the Dirac equation itself, from the zitter model. But that possibility will not be explored here.

The zitter Dirac equation offers a new perspective on the significance of negative energy in Dirac theory. Recall that negative energy solutions were first regarded as a serious defect of the Dirac equation. Schroedinger showed that they are essential for a general solution even in the free particle case when he "discovered zitterbewegung" as interference between positive and negative components of a wave packet. In an audacious effort to save the theory, Dirac identified the negative energy states with an "anti-electron" and invoked the Pauli principle to suppress them (hole theory). Miraculously, the positron was discovered shortly thereafter, so the defect was transmuted to a spectacular triumph!

A standard conclusion from all this is expressed by the following quotations [34]: "The zitterbewegung demonstrates in a real sense a single particle theory is not possible." "The difficulties with the negative energy states of the Dirac equation almost of necessity demand a many-body theory." "*Hole theory is a many-body theory* describing particles with positive and negative charge. The simple probability interpretations of the wave functions acclaimed in a single-particle theory cannot be true any longer, because the creation and annihilation of electron-positron pairs must be taken in account in the wave function." In other words, quantum field theory is needed to explain zitterbewegung!

For a different perspective on negative energy we note that $\psi^C \equiv \psi\boldsymbol{\sigma}_2$ is the charge conjugate solution of the real Dirac equation [1]. Hence we can cast the zitter wave function (248) in the form

$$\psi_+ \equiv \tfrac{1}{2}(\psi + \psi^C). \qquad (256)$$

This expresses Dirac's negative energy solution as an essential component of the zitter rather than an antiparticle. It is an alternative splitting of Dirac's 4-component wave function into a pair of 2-component wave functions for different particle states. The physical issue is this: Which components of the Dirac wave function should be identified with the electron? The zitter component ψ_+ describes an electron with zitter motion. Standard quantum field theory splits the zitter into positive and negative energy components and then reassembles it later from pair creation and annihilation. In most calculations the end result will be the same, because both approaches start from the same Dirac equation. Contrary to the standard Dirac equation (224), the zitter Dirac equation (249) is consistent with a single particle model of zitter without the strenuous expedient of field quantization. This is not to deny that some version of quantum field theory is necessary to account for creation and annihilation of particles. The problem is to devise experiments that identify the basic particle states.

D. Electroweak interactions

We have seen that only half the Dirac wave function $\psi_+ = \psi\frac{1}{2}(1+\boldsymbol{\sigma}_2)$ is needed to describe the electron. What can be said about the other half $\psi_- = \psi\frac{1}{2}(1 - \boldsymbol{\sigma}_2)$? An attractive answer is suggested by gauge theory.

In the real Dirac equation (224) an electromagnetic gauge transformation of the wave function is multiplicative on the right with the form

$$\psi \quad \rightarrow \quad \psi' = \psi U, \qquad (257)$$

where $U = \exp{(i\boldsymbol{\sigma}_3\chi)}$. We look to adapt this transformation to the zitter Dirac equation (249) and generalize it in a way that preserves essential structure of the equation. To preserve the mass term, we require

$$\tilde{U}\gamma_0 U = \gamma_0. \qquad (258)$$

The general solution of this equation has the form

$$U = e^{\frac{1}{2}i\boldsymbol{\theta}}e^{\frac{1}{2}i\chi}, \qquad (259)$$

where $\boldsymbol{\theta} = \theta_1\boldsymbol{\sigma}_1 + \theta_2\boldsymbol{\sigma}_2 + \theta_3\boldsymbol{\sigma}_3$. Remarkably, this is the gauge group SU(2)⊗U(1) of electroweak theory. It strongly suggests that the geometric structure of electroweak theory is already inherent in Dirac theory! Indeed, it requires that we identify $\psi_- \equiv \psi_\nu$ with the neutrino, just as we identify $\psi_+ \equiv \psi_e$ with the electron.

Thus, we construe the Dirac wave function as a lepton wave function composed of electron and neutrino components:

$$\psi = \psi\tfrac{1}{2}(1 + \boldsymbol{\sigma}_2) + \psi\tfrac{1}{2}(1 - \boldsymbol{\sigma}_2) = \psi_e + \psi_\nu. \qquad (260)$$

To adapt the Dirac equation to this interpretation, we follow standard electroweak theory in introducing a gauge covariant derivative

$$\mathcal{D}_\mu\psi \equiv \partial_\mu\psi - \psi i W_\mu$$
$$\text{with} \qquad W_\mu = \frac{1}{2\hbar}(g\mathbf{A}_\mu - g'B_\mu). \qquad (261)$$

In the standard way, gauge covariance is assured by requiring that the electroweak connexion satisfies the transformation law

$$W_\mu \quad \rightarrow \quad W'_\mu = \tilde{U}W_\mu U - \tilde{U}\partial_\mu U. \qquad (262)$$

Accordingly, the Dirac equation generalizes to the lepton wave equation

$$\gamma^\mu\mathcal{D}_\mu\psi i\boldsymbol{\sigma}_3\hbar = m_e\psi_e\gamma_0. \qquad (263)$$

For the electron component, this reduces to the zitter Dirac equation (249) when weak interactions are turned off.

Details of this model integrating zitterbewegung with electroweak theory are discussed elsewhere [30]. It is amusing to recall that one of the early suggestions to account for extra components in the Dirac wave function was to identify them as proton states. Long dismissed because the particles had different mass, the idea returns again as an electron-neutrino wave function.

X. CONCLUSIONS

The zitterbewegung, if it turns out to be physically real, is belated confirmation of de Broglie's original hypothesis [36] that the electron has an internal clock with period precisely equal to twice the zitter period, precisely the relation between the period of a rotor and that of a vector it rotates.

As we have seen, the physical signature of zitter is a rotating electric dipole with ultra high frequency. If this exists, its implications for quantum mechanics will be far-reaching. Evidently it can be incorporated in Dirac theory by subtle changes in the specification of observables and the structure of the Dirac equation.

Experimental confirmation of the zitter should stimulate research on its proposed incorporation into electroweak theory. Then study of zitter self-interaction should look for excitations explaining the three lepton families. Finally, the strong analogy between electroweak interactions of leptons and quarks suggests that one should investigate modifications of zitter structure to model quarks and strong interactions. All this should go hand-in-hand with development of zitter field theory and reconciliation of it with quantum field theory.

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