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A matrix generalization of Euler identity $e^{j\varphi} = \cos\varphi + j \sin\varphi$

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Abstract

In this work we present a matrix generalization of the Euler identity about exponential representation of a complex number. The concept of matrix exponential is used in a fundamental way. We define a notion of matrix imaginary unit which generalizes the usual complex imaginary unit. The Euler-like identity so obtained is compatible with the classical one. Also, we derive some exponential representation for matrix real and imaginary unit, and for the first Pauli matrix.

Keywords: Euler identity, matrix exponential, series expansion, matrix unit representation, Pauli matrices representation.

1 The matrix exponential

Let \mathbf{A} denotes a generic matrix. Based on the Taylor expansion centered at 0 for the one real variable function e^x , the *matrix exponential* (see e.g. [3]) is formally defined as

$$e^{\mathbf{A}} = \sum_{n=0}^{+\infty} \frac{\mathbf{A}^n}{n!} \quad (1)$$

2 A class of complex matrices

Let it be j the imaginary unit, $j^2 = -1$, α , a , b real numbers, $\alpha \neq 0$. We consider the class of 2×2 matrices of the form

$$\mathbf{T} = \begin{pmatrix} a & jb \\ j\alpha^2 b & a \end{pmatrix}$$

These matrices are very important in the theory of transfer matrix method for modelization of acoustical transmission in physical structures (see [5], [2]). Note that $\text{Det}(\mathbf{T}) = a^2 + \alpha^2 b^2$, so that if a and b are not both zero, \mathbf{T} is invertible. In acoustical transfer matrices, usually $\alpha = \frac{S}{c}$, where S is the section of a tube or duct and c is the sound speed in the fluid contained in the tube.

Consider the matrix

$$\mathbf{\Phi} = \begin{pmatrix} 0 & j \\ j\alpha^2 & 0 \end{pmatrix}$$

Then, if \mathbf{I} is the identity matrix, the following representation for previous \mathbf{T} matrices holds:

$$\mathbf{T} = a\mathbf{I} + b\mathbf{\Phi} \tag{2}$$

Note the analogy with a usual complex number $a + jb$. The analogy is more evident if one consider that, with a simple calculation, $\mathbf{\Phi}^2 = -\alpha^2\mathbf{I}$. For this reason, we call $\mathbf{\Phi}$ the *imaginary unit matrix*. Also, note that, for $\alpha = 1$, we obtain $\mathbf{\Phi} = j\sigma_1$, where σ_1 is one of the *Pauli matrices* of quantum mechanics (see e.g. [4]).

3 A generalization of Euler identity

The *Euler* identity $e^{j\varphi} = \cos\varphi + j \sin\varphi$ is valid for any real φ . Usually this formula is proven by use of Taylor expansion of the complex function e^z and of the real functions $\cos\varphi$ and $\sin\varphi$ (see [1]).

We prove a generalization, in the environment of the complex matrices of type (2), of this identity.

Lemma 1 *Let it be $\Psi = -j\Phi$. Then, for every natural n , the following relation holds:*

$$\Psi^n = \alpha^{n-r} \Psi^r \quad (3)$$

where $r = \text{mod}(n, 2)$.

Dim. By induction on n . For $n = 0$ and $n = 1$ the thesis is obvious. Note that

$$\Psi = \begin{pmatrix} 0 & 1 \\ \alpha^2 & 0 \end{pmatrix}$$

Let it be $n = 2$. By a simple calculation

$$\Psi^2 = \begin{pmatrix} 0 & 1 \\ \alpha^2 & 0 \end{pmatrix}^2 = \begin{pmatrix} \alpha^2 & 0 \\ 0 & \alpha^2 \end{pmatrix} = \alpha^2 \mathbf{I} = \alpha^2 \Psi^0$$

and the thesis is verified. Then we suppose that the thesis is verified for a generic n too. Using the last inductive step we have

$$\Psi^{n+1} = \Psi^n \Psi = \alpha^{n-r} \Psi^r \Psi \quad (4)$$

where $r = \text{mod}(n, 2)$. If $\text{mod}(n+1, 2) = 1$, then $r = 0$, therefore

$$\Psi^{n+1} = \alpha^n \Psi = \alpha^{n+1-s} \Psi^s \quad (5)$$

with $s = \text{mod}(n+1, 2) = 1$, and the thesis is true in this case. If $\text{mod}(n+1, 2) = 0$, then $r = 1$, therefore, using the first inductive step for $n = 2$,

$$\Psi^{n+1} = \alpha^{n-1} \Psi \Psi = \alpha^{n+1} \Psi^0 = \alpha^{n+1-s} \Psi^s \quad (6)$$

with $s = \text{mod}(n+1, 2) = 0$, and the thesis is true in this case too. \square

Now we can prove the matrix generalization of Euler identity:

Theorem 1 *For every real φ , the following formula holds:*

$$e^{\varphi \Phi} = \cos(\alpha \varphi) \mathbf{I} + \frac{1}{\alpha} \sin(\alpha \varphi) \Phi \quad (7)$$

Dim. If $\varphi = 0$ the formula is obvious. For $\varphi \neq 0$, from the formal definition (1) we can write

$$e^{\varphi\Phi} = \sum_{n=0}^{+\infty} \frac{(\varphi\Phi)^n}{n!} = \sum_{n \text{ odd}} \frac{\varphi^n \Phi^n}{n!} + \sum_{n \text{ even}} \frac{\varphi^n \Phi^n}{n!} \quad (8)$$

Recall that, if n is even, then j^n alternates -1 and $+1$, while if n is odd, then j^n alternates $-j$ and $+j$. Therefore, from $\Phi = j\Psi$, from the usual series expansion for $\cos(\alpha\varphi)$ and $\sin(\alpha\varphi)$, and using the previous Lemma, we have

$$\begin{aligned} e^{\varphi\Phi} &= \sum_{n \text{ even}} \frac{j^n \varphi^n \Psi^n}{n!} + \sum_{n \text{ odd}} \frac{j^n \varphi^n \Psi^n}{n!} = \\ &= \left(\sum_{n \text{ even}} j^n \frac{(\alpha\varphi)^n}{n!} \right) \mathbf{I} + \frac{1}{\alpha} \left(\sum_{n \text{ odd}} j^n \frac{(\alpha\varphi)^n}{n!} \right) \Psi = \\ &= \cos(\alpha\varphi) \mathbf{I} + \frac{1}{\alpha} j \sin(\alpha\varphi) \Psi = \cos(\alpha\varphi) \mathbf{I} + \frac{1}{\alpha} \sin(\alpha\varphi) \Phi \end{aligned} \quad (9)$$

that is the thesis. \square

Note 1. Let it be $\alpha = 1$, and $\mathbf{I} = [1]$, $\Phi = [j]$ two 1×1 matrices, so that \mathbf{I} is the usual real unit and Φ the usual imaginary unit. From (7) we have

$$e^{j\varphi} = \cos(\varphi)[1] + \sin(\varphi)[j] = \cos\varphi + j \sin\varphi \quad (10)$$

that is the classical Euler identity.

Note 2. If we write in explicite mode the relation (7), we obtain

$$e^{\varphi\Phi} = \begin{pmatrix} \cos(\alpha\varphi) & j\frac{1}{\alpha}\sin(\alpha\varphi) \\ j\alpha \sin(\alpha\varphi) & \cos(\alpha\varphi) \end{pmatrix}$$

so that $\text{Det}(e^{\varphi\Phi}) = 1$, which is compatible with the fact that for usual complex numbers $|e^{j\varphi}| = 1$.

Note 3. If $\alpha = 1$ and $\varphi = 2m\pi$, with m integer, (7) becomes

$$e^{2m\pi\Phi} = \mathbf{I} \quad (11)$$

that is a matrix unit representation. The classical analogous formula is $e^{j2m\pi} = 1$.

Note 4. If $\alpha = 1$ and $\varphi = m\frac{\pi}{2}$, with $m = 1 + 4k$, k integer, (7) becomes

$$e^{m\frac{\pi}{2}\Phi} = \Phi \quad (12)$$

that is a matrix imaginary unit representation. The classical analogous formula is $e^{jm\frac{\pi}{2}} = j$. Also, if we multiply previous formula by $-j$, we have an exponential representation of Pauli matrix σ_1 :

$$\sigma_1 = -je^{m\frac{\pi}{2}\Phi} \quad (13)$$

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