3D Hessian

For 3D elastic equation, we can express it as

$$
\begin{bmatrix}
(\lambda+2\mu)\frac{\partial}{\partial x} & \lambda\frac{\partial}{\partial y} & \lambda\frac{\partial}{\partial z} & \frac{\partial}{\partial t} & & & & & \\
\lambda\frac{\partial}{\partial x} & (\lambda+2\mu)\frac{\partial}{\partial y} & \lambda\frac{\partial}{\partial z} & & \frac{\partial}{\partial t} & & & & \\
\lambda\frac{\partial}{\partial x} & \lambda\frac{\partial}{\partial y} & (\lambda+2\mu)\frac{\partial}{\partial z} & & & \frac{\partial}{\partial t} & & & \\
\mu\frac{\partial}{\partial y} & \mu\frac{\partial}{\partial x} & & & & & \frac{\partial}{\partial t} & & \\
\mu\frac{\partial}{\partial z} & & \mu\frac{\partial}{\partial x} & & & & & \frac{\partial}{\partial t} & \\
 & \mu\frac{\partial}{\partial z} & \mu\frac{\partial}{\partial y} & & & & & & \frac{\partial}{\partial t} \\
\rho\frac{\partial}{\partial t} & & & \frac{\partial}{\partial x} & & & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \\
 & \rho\frac{\partial}{\partial t} & & & \frac{\partial}{\partial y} & & \frac{\partial}{\partial x} & & \frac{\partial}{\partial z} \\
 & & \rho\frac{\partial}{\partial t} & & & \frac{\partial}{\partial z} & & \frac{\partial}{\partial x} & \frac{\partial}{\partial y}
\end{bmatrix}
\begin{bmatrix} v_x \\ v_y \\ v_z \\ \tau_{xx} \\ \tau_{yy} \\ \tau_{zz} \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix}
=
\begin{bmatrix} s_x \\ s_y \\ s_z \\ f_{xx} \\ f_{yy} \\ f_{zz} \\ f_{xy} \\ f_{xz} \\ f_{yz} \end{bmatrix}
\tag{1}
$$

For simplification, we have Eq. (1) in a matrix form: $\mathbf{Ad}=\mathbf{f}$.

When it comes to parameterization of (α, β, ρ),

$$\frac{\partial \mathbf{A}}{\partial \alpha} = \begin{bmatrix} 2\rho\alpha\frac{\partial}{\partial x} & 2\rho\alpha\frac{\partial}{\partial y} & 2\rho\alpha\frac{\partial}{\partial z} & 0 & & & & & \\ 2\rho\alpha\frac{\partial}{\partial x} & 2\rho\alpha\frac{\partial}{\partial y} & 2\rho\alpha\frac{\partial}{\partial z} & & 0 & & & & \\ 2\rho\alpha\frac{\partial}{\partial x} & 2\rho\alpha\frac{\partial}{\partial y} & 2\rho\alpha\frac{\partial}{\partial z} & & & 0 & & & \\ & & & & & & 0 & & \\ & & & & & & & 0 & \\ & & & & & & & & 0 \end{bmatrix}$$

$$\frac{\partial \mathbf{A}}{\partial \beta} = \begin{bmatrix} 0 & -4\rho\beta\frac{\partial}{\partial y} & -4\rho\beta\frac{\partial}{\partial z} & 0 & & & & & \\ -4\rho\beta\frac{\partial}{\partial x} & 0 & -4\rho\beta\frac{\partial}{\partial z} & & 0 & & & & \\ -4\rho\beta\frac{\partial}{\partial x} & -4\rho\beta\frac{\partial}{\partial y} & 0 & & & 0 & & & \\ 2\rho\beta\frac{\partial}{\partial y} & 2\rho\beta\frac{\partial}{\partial x} & & & & & 0 & & \\ 2\rho\beta\frac{\partial}{\partial z} & & 2\rho\beta\frac{\partial}{\partial x} & & & & & 0 & \\ & 2\rho\beta\frac{\partial}{\partial z} & 2\rho\beta\frac{\partial}{\partial y} & & & & & & 0 \end{bmatrix}$$

$$
\frac{\partial \mathbf{A}}{\partial \alpha}\mathbf{d} =
\begin{bmatrix}
2\rho\alpha\frac{\partial}{\partial x} & 2\rho\alpha\frac{\partial}{\partial y} & 2\rho\alpha\frac{\partial}{\partial z} & 0 & & & & & \\
2\rho\alpha\frac{\partial}{\partial x} & 2\rho\alpha\frac{\partial}{\partial y} & 2\rho\alpha\frac{\partial}{\partial z} & & 0 & & & & \\
2\rho\alpha\frac{\partial}{\partial x} & 2\rho\alpha\frac{\partial}{\partial y} & 2\rho\alpha\frac{\partial}{\partial z} & & & 0 & & & \\
 & & & & & & 0 & & \\
 & & & & & & & 0 & \\
 & & & & & & & & 0 \\
 & & & & & & & & \\
 & & & & & & & & \\
 & & & & & & & &
\end{bmatrix}
\begin{bmatrix} v_x \\ v_y \\ v_z \\ \tau_{xx} \\ \tau_{yy} \\ \tau_{zz} \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix}
=
\begin{bmatrix}
2\rho\alpha(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}) \\
2\rho\alpha(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}) \\
2\rho\alpha(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}) \\
 \\
 \\
 \\
 \\
 \\
 \\
\end{bmatrix}
$$

$$
\frac{\partial \mathbf{A}}{\partial \beta}\mathbf{d} =
\begin{bmatrix}
0 & -4\rho\beta\frac{\partial}{\partial y} & -4\rho\beta\frac{\partial}{\partial z} & 0 & & & & & \\
-4\rho\beta\frac{\partial}{\partial x} & 0 & -4\rho\beta\frac{\partial}{\partial z} & & 0 & & & & \\
-4\rho\beta\frac{\partial}{\partial x} & -4\rho\beta\frac{\partial}{\partial y} & 0 & & & 0 & & & \\
2\rho\beta\frac{\partial}{\partial y} & 2\rho\beta\frac{\partial}{\partial x} & & & & & 0 & & \\
2\rho\beta\frac{\partial}{\partial z} & & 2\rho\beta\frac{\partial}{\partial x} & & & & & 0 & \\
 & 2\rho\beta\frac{\partial}{\partial z} & 2\rho\beta\frac{\partial}{\partial y} & & & & & & 0 \\
 & & & & & & & & \\
 & & & & & & & & \\
 & & & & & & & &
\end{bmatrix}
\begin{bmatrix} v_x \\ v_y \\ v_z \\ \tau_{xx} \\ \tau_{yy} \\ \tau_{zz} \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix}
=
\begin{bmatrix}
-4\rho\beta(\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}) \\
-4\rho\beta(\frac{\partial v_x}{\partial x}+\frac{\partial v_z}{\partial z}) \\
-4\rho\beta(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}) \\
2\rho\beta(\frac{\partial v_x}{\partial y}+\frac{\partial v_y}{\partial x}) \\
2\rho\beta(\frac{\partial v_x}{\partial z}+\frac{\partial v_z}{\partial x}) \\
2\rho\beta(\frac{\partial v_y}{\partial z}+\frac{\partial v_z}{\partial y}) \\
 \\
 \\
 \\
\end{bmatrix}
$$

Finally, for source-source and source-receiver combinations, we have, respectively, as

SS: $H_{\alpha\alpha} = \sum \left(\frac{\partial \mathbf{A}}{\partial \alpha} \mathbf{d} \right) \left(\frac{\partial \mathbf{A}}{\partial \alpha} \mathbf{d} \right)^{*}, \quad H_{\beta\beta} = \sum \left(\frac{\partial \mathbf{A}}{\partial \beta} \mathbf{d} \right) \left(\frac{\partial \mathbf{A}}{\partial \beta} \mathbf{d} \right)^{*}$

SR: $H_{\alpha\alpha} = \sum \left| \left(\frac{\partial \mathbf{A}}{\partial \alpha} \mathbf{d} \right) \left(\frac{\partial \mathbf{A}}{\partial \alpha} \tilde{\mathbf{d}} \right)^{*} \right|, \quad H_{\beta\beta} = \sum \left| \left(\frac{\partial \mathbf{A}}{\partial \beta} \mathbf{d} \right) \left(\frac{\partial \mathbf{A}}{\partial \beta} \tilde{\mathbf{d}} \right)^{*} \right|$