Expert Systems With Applications 147 (2020) 113222

Contents lists available at ScienceDirect

Expert Systems With Applications

journal homepage: www.elsevier.com/locate/eswa

Topological data analysis in investment decisions

Anubha Goel[a,*], Puneet Pasricha[b], Aparna Mehra[a]

[a] Department of Mathematics, Indian Institute of Technology Delhi, Hauz Khas, New Delhi 110016, India
[b] School of Mathematics and Applied Statistics, University of Wollongong, Northfield Avenue, New South Wales 2500, Australia

ARTICLE INFO

Article history:
Received 18 August 2019
Revised 11 January 2020
Accepted 18 January 2020
Available online 20 January 2020

Keywords:
Topological data analysis
Persistence landscape
Takens' embedding
Portfolio selection
Enhanced indexing

ABSTRACT

This article explores the applications of Topological Data Analysis (TDA) in the finance field, especially addressing the primordial problem of asset allocation. Firstly, we build a rationale on why TDA can be a better alternative to traditional risk indicators such as standard deviation using real data sets. We apply Takens embedding theorem to reconstruct the time series of returns in a high dimensional space. We adopt the sliding window approach to draw the time-dependent point cloud data sets and associate a topological space with them. We then apply the persistent homology to discover the topological patterns that appear in the multidimensional time series. The temporal changes in the persistence landscapes, which are the real-valued functions that encode the persistence of topological patterns, are captured via L^p norm. The time series of the L^p norms shows that it is better at measuring the dynamics of returns than the standard deviation.

Inspired by our findings, we explore an application of TDA in Enhanced Indexing (EI) that aims to build a portfolio of fewer assets than that in the index to outperform the latter. We propose a two-step procedure to accomplish this task. In step one, we utilize the L^p norms of the assets to propose a filtration technique of selecting a few assets from a larger pool of assets. In step two, we propose an optimization model to construct an optimal portfolio from the class of filtered assets for EI. To test the efficiency of this enhanced algorithm, experiments are carried out on ten data sets from financial markets across the globe. Our extensive empirical analysis exhibits that the proposed strategy delivers superior performance on several measures, including excess mean returns from the benchmark index and tail reward-risk ratios than some of the existing models of EI in the literature. The proposed filtering strategy is also noted to be beneficial for both risk-seeking and risk-averse investors.

1. Introduction

Financial markets are highly competitive and information-driven, where any piece of supplementary information could be of significant value to an investor. Understanding both the quantitative and qualitative properties of financial data is of immense importance. To this context, academicians and professionals suggested various modeling techniques. For instance, an increasing number of studies applied advanced machine learning and deep learning models in forecasting. However, the statistical methods are not adequate to handle the growing complexity of data, more specifically, the qualitative properties such as shapes and structures present in the data (Munch, 2017; Umeda, Kaneko, & Kikuchi, 2019).

TDA is one of the emerging areas that uses tools from algebraic topology to discover structures in the data and provides novel and useful insights that the conventional data analysis techniques may not be able to capture (Chazal et al., 2017). The underlying idea in TDA is that data has shape, shape has meaning and meaning drives values. The modern-day data usually appear as point clouds embedded in high dimensional Euclidean space or general metric space. These point clouds are not uniformly distributed but possess highly non-linear geometries with non-trivial topology. TDA helps extract topological information from the point clouds and summarizes it in the persistence diagram. An alternative persistence landscape tool is used to convey the information contained in the persistence diagram. A persistence landscape comprises of a sequence of continuous and piecewise linear functions defined on a re-scaled birth-death coordinate. These landscapes are embedded in a Banach space, unlike persistence diagrams, which have a natural metric space structure. Both these tools are robust under data perturbations.

* Corresponding author.
E-mail addresses: anubha.goel1@gmail.com (A. Goel), puneetp@uow.edu.au (P. Pasricha), apmehra@maths.iitd.ac.in (A. Mehra).

https://doi.org/10.1016/j.eswa.2020.113222
0957-4174/

Though some information would be lost while summarizing the shapes and structures using TDA, the structure in the data would remain preserved, unlike the traditional statistical techniques that lose on the vital structure present in the data. The salient feature of TDA is that it is free from any apriori assumption on the distribution of the data. Pereira and de Mello (2015) used an artificial example to demonstrate the insights that TDA can provide, which other statistical methods may fail to capture. They considered three-time series $S_1 = \sin(2\pi t)$, $S_2 = e^{-t}\cos(2\pi t)$ and $S_3 = \sin\left(4\pi t + \frac{\pi}{2}\right)$, where S_1 and S_3 are periodic sine waves with different frequencies while S_2 is a damped sine wave. Applying dynamic time wrapping (DTW), they observed that S_1 and S_2 are together closer in the distance than S_1 and S_3, even though S_1 and S_3 are both periodic functions, while S_2 is not. The reason is that high frequency of S_3 increases the distance measure between S_1 and S_3 resulting in the observed behavior. They after that, applied persistent homology (a concept in TDA) to quantify the topological features in three-time series and observed that S_1 and S_3 have the same topological behavior. Through this experiment, they demonstrated that TDA could capture the qualitative features of data in contrast to the statistical-based methods, like DTW, that focus on quantitative aspects. This phenomenon is also observed by Ravishanker and Chen (2019) in their very recent research article.

Recent advances in TDA have shown great promise in extracting information on the topological features of data and investigate their properties. The pioneer contributions of Edelsbrunner, Letscher, and Zomorodian (2000) and Zomorodian and Carlsson (2005) laid the foundation of TDA. An overview paper by Carlsson (2009) turns out to be cornerstone. Before this, a few studies exist in the literature that utilizes topological ideas to analyze financial data, but these methods primarily are based on geometry and network reconstruction. For instance, Vandewalle et al. (2001) studied the correlation structures among the USA stocks using a minimum spanning tree and analyzed the topology exhibited by the minimum spanning tree. Phoa (2013) used diffusion maps to project stock market data to a 3-D hyperplane and studied the correlations structures by observing distances between stocks in the hyperplane. However, such studies are scarce.

Persistent homology is applied successfully in a large number of problems in fields as diverse as bioinformatics, medicine, neuroscience, astrophysics, and social sciences. The list is long to enumerate here but to cite a few (Berwald, Gidea, & Vejdemo-Johansson, 2013; Carlsson, Ishkhanov, De Silva, & Zomorodian, 2008; De Silva & Ghrist, 2007a; De Silva & Ghrist, 2007b; Emrani, Gentimis, & Krim, 2014; Freedman & Chen, 2009; Heo, Gamble, & Kim, 2012; Kasson et al., 2007; Khasawneh & Munch, 2016; Kovacev-Nikolic, Bubenik, Nikolić, & Heo, 2016; Maletić, Zhao, & Rajković, 2016; Munch, 2017; Nicolau, Levine, & Carlsson, 2011; Perea, Deckard, Haase, & Harer, 2015; Perea & Harer, 2015; Pereira & de Mello, 2015; Piangerelli, Rucco, Tesei, & Merelli, 2018; Seversky, Davis, & Berger, 2016; Singh et al., 2008; Truong, 0000; Zhu, 2013).

Several articles (Chazal, Glisse, Labruere, & Michel, 2013; Fasy et al., 2014; Mileyko, Mukherjee, & Harer, 2011; 2011; Munch et al., 2015; Turner, Mileyko, Mukherjee, & Harer, 2014) developed statistical methods to model and analyze multiple persistence diagrams. Mileyko et al. (2011) studied the probabilistic basis of random persistence diagrams. Blumberg, Gal, Mandell, and Pancia (2014) provided statistical approach to TDA. Chazal, Fasy, Lecci, Rinaldo, and Wasserman (2014) applied bootstrapping to the persistence landscape. Taylor et al. (2015) applied TDA to study spread of contagions in networks.

Gidea (2017) used TDA to detect early warning signs of crisis transitions in financial markets. They referred to critical transition as an abrupt change in market behavior due to small perturbations in external conditions, which cause the system to shift from one stable state to the other. Using the price data of Dow-Jones Industrial Average stocks from January 2004 to September 2008, they constructed a weighted network by correlations distances. They applied the persistent homology to this network to quantify changes in the structure of data towards the end of the period. Significant topological changes were observed in the networks just before the financial crisis. In another work, Gidea and Katz (2018) followed a similar approach to explore applications of TDA on the technology crash of 2000 and the sub-prime crisis of 2008. The authors used the sliding window approach to obtain point clouds from data of stock prices. Subsequently, persistent homology was used to extract topological features encoded in the persistence landscapes. The temporal changes in landscapes were quantified using L^p norms. They observed a sharp rise in the L^p norms just about 250 days before each of the financial crashes, thus demonstrating the capability of TDA in generating early warning signals of the financial crisis. These two studies demonstrate that TDA could be a promising area to analyze financial data.

Recently, researchers amalgamated machine learning and persistent homology. Pun, Xia, and Lee (2018) presented a survey of persistent homology and presented supervised and unsupervised models. The success of TDA in various fields raised our curiosity to investigate if the TDA methodology can deliver the same success on financial data. More so, the recent review article by Ravishanker and Chen (2019) on TDA application in time-series and the exciting findings by Makridakis, Spiliotis, and Assimakopoulos (2019) on the outcomes from M4 competition that in contrast to the single method a combination of methods is more accurate in capturing time-series because it cancels the errors of the individual models through averaging. We believe that the novel and relatively incipient field of TDA can shed more light on the topological features in financial data, which in turn can benefit investment decisions.

In this paper, we aim to devise a new method for portfolio construction in the context of enhanced indexing (EI) using tools from TDA. The proposed method operates in two stages: filtering of assets and solving an optimization problem on filtered assets. Firstly, we propose a strategy to identify a few stocks capable of outperforming the index. This task is accomplished by applying Takens' embedding theorem, fitting a persistence landscape, and computing the L^p-norms. We divided the whole class of assets into three bins based on their L^p norm values and studied their properties. In step two, we propose an optimization model, similar to the mean absolute deviation model, to construct a portfolio that tracks the index volatility to minimize risk with a constraint for achieving return over and above the index. We applied the model on the data of the filtered assets only. In the empirical analysis, we consider daily prices of ten data sets from markets across the globe, covering an extended period. The performance of the proposed model is shown to outperform on excess mean return and reward-risk ratios compared to the other three well-known existing models of enhanced indexing in the literature.

1.1. Enhanced indexing

Portfolio construction is the most recurrent financial problem. It is always a challenge for investment managers to construct diversified portfolios that beat the benchmark index. This primordial problem has attracted attention under the name of enhanced indexing (EI) or enhanced index tracking (EIT) (see, Canakgoz & Beasley, 2009; Goel, Sharma, & Mehra, 2018a, and references therein).

The studies by DiBartolomeo (2000) and Riepe and Werner (1998) are among the earliest to discuss EI funds. Ahmed and Nanda (2005) reported the growth of the EI funds over the span of 20 years where most of such funds target the S&P 500 index. Koshizuka, Konno, and Yamamoto (2009) and Weng and Wang (2017) reported an increasing popularity of the EI funds in Tokyo and China markets, respectively. Koshizuka et al. (2009) proposed an optimization model using mean absolute deviation (MAD) and semi MAD to find an index-plus-alpha portfolio[1] among all those portfolios which are highly correlated to the index.

Canakgoz and Beasley (2009) proposed a bi-objective mixed linear EI model where alpha of the portfolio is maximized and beta of the portfolio about unity is minimized.

Li, Sun, and Bao (2011) formulated the bi-objective programming problem of maximizing the excess mean return and minimizing the downside deviation of order two from index return along with transaction cost constraints. An immune based multi-objective optimization algorithm is employed to find a solution to the problem.

Filippi, Guastaroba, and Speranza (2016) applied a heuristic approach which combines the kernel search with the ϵ-constraint method to solve the integer linear bi-objective programming problem of maximizing the excess mean return and minimizing the absolute deviation between returns of portfolio and index with real-life features.

Bruni, Cesarone, Scozzari, and Tardella (2015) proposed a linear program for EI by maximizing excess mean return and imposed a bound on the worst performance of portfolio return from the index, originally described in Rudolf, Wolter, and Zimmermann (1999). Bruni, Cesarone, Scozzari, and Tardella (2017) proposed cumulative zero-order ϵ-SD in EI.

Paulo, de Oliveira, and do Valle Costa (2016) proposed an optimization model by creating a trade-off between the weighted sum of excess mean return and variance of the difference of returns of portfolio and index. They allowed short-selling and hence obtained the close form solution for the optimal weights when the assets for investment are prior selected.

More extensive reviews of this problem can be found in Beasley, Meade, and Chang (2003), Canakgoz and Beasley (2009), Scozzari, Tardella, Paterlini, and Krink (2013) and, more recently, in Goel et al. (2018a) and Goel and Sharma (2019).

1.2. Flow of the article

Section 2 explains the basic concepts of TDA. Section 3 describes the sample data and proposes a methodology for empirical analysis. Section 4 analyzes the significance of L^p norm values in financial data. Section 5 introduces an optimization model for enhanced indexing and presents the experimental results along with a comparative analysis with some of the existing EI models relevant to the context. Section 6 concludes the paper with some directions for future research.

2. Background

This section includes a review of a few basic notions from algebraic topology to build the theoretical background of TDA. The textbook by Hatcher (2005) is an excellent source to begin.

2.1. Simplicial complex and homology

A k-simplex is the convex hull of $k+1$ affinely independent points $\{a_0, a_1, \ldots, a_k\} \subset R^q$. A simplicial complex S is a finite set of simplices that satisfy two essential conditions: (i) any face of a simplex from S is also in S, that is, if $\omega \in S$ and $\sigma \subset \omega$ then $\sigma \in S$;

(ii) intersection of any two simplices in S is either empty or shares faces.

A kth chain group $C_k(S)$ of a simplicial complex S is an Abelian group made from all k-chains from the simplicial complex S together with the addition operation. Define a linear boundary operator $\partial_k : C_k(S) \mapsto C_{k-1}(S)$ as $\partial_k([\sigma^k]) = \sum_{i=0}^{k}[a_0, a_1, \ldots, \hat{a}_i, \ldots, a_k]$, where $[\sigma^k] = [a_0, a_1, \ldots, a_i, \ldots a_k]$ denotes an oriented k-simplex and $[a_0, a_1, \ldots, \hat{a}_i, \ldots, a_k]$ is a $k-1$ oriented simplex generated by $\{a_0, a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_k\}$. Also, if $k=0$, then we define $\partial_0([\sigma^0]) = 0$.

The collection $\{C_k(S), \partial_k\}_{k \geq 1}$ is known as chain complex of S. One can think of the linear map ∂_k as algebraically capturing the $(k-1)$-dimensional boundary of a k-dimensional object. Therefore, we have $\partial_{k-1} \circ \partial_k = 0$ i. e., the boundary of a boundary does not exist. We call an k-chain a cycle if its boundary is zero. As a result of which, the set of all k cycles $Z_k = \text{Ker } \partial_k = \{\xi \in C_k(S) \,|\, \partial_k \xi = 0\}$ and the set $B_k = \text{Im } \partial_{k+1} = \{\xi \in C_k(S) \,|\, \xi = \partial_{k+1} c, \text{ for some } c \in \partial C_{k+1}\}$ form subgroups of $C_k(S)$. The quotient group $H_k = Z_k / B_k$ is called the k-th homology group. The rank of H_k is called k-th Betti number β_k. Thus, we can regard β_0 as the number of isolated components, β_1 is the number of one dimensional loops or circles, β_2 are the number of two-dimensional voids, cavities or holes. In general, β_i counts the number of i-th dimensional holes in S. For a more detailed discussion on persistent homology, one can refer to Carlsson (2009), Edelsbrunner et al. (2000), Otter, Porter, Tillmann, Grindrod, and Harrington (2017), and Pun et al. (2018).

2.2. Vietoris–Rips complex and persistence diagrams

Let P be a point set in $\mathbb{R}^q$. Let $\mathcal{R} = \{R(P, \epsilon),\ \epsilon > 0\}$ be a (right continuous) filtration of simplicial complexes i. e., $R(P, \epsilon_1)$ is a subcomplex of $R(P, \epsilon_2)$ for $\epsilon_1 < \epsilon_2$ and $R(P, \epsilon) = \cap_{\epsilon < r} R(P, r)$. Although there are several methods to obtain a filtration of simplicial complexes, we adopt the Vietoris–Rips filtration (Ghrist, 2008), a brief idea of which is presented below.

A Vietoris–Rips complex (or simply Rips complex)$R(P, \epsilon)$, with a pre-specified threshold $\epsilon > 0$, is the set of all k-simplices of P such that the largest Euclidean distance between any of its vertices is at most 2ϵ.

A step-wise procedure for constructing a Rips complex is described below:

1. Connect each pair of points $a_1, a_2 \in P$, having distance $d(a_1, a_2) < 2\epsilon$, with an edge.
2. Connect each triplet $a_1, a_2, a_3 \in P$ such that the distance $d(a_i, a_j) < 2\epsilon, \forall\, 1 \leq i, j \leq 3$, to form a triangle.
3. Connect each set of p-points as a $p-1$ dimensional simplex if all of them are pairwise within a distance of 2ϵ.

Fig. 1 illustrates the process, and Algorithm 1 summarizes the step-wise procedure.

Algorithm 1 Construction of Vietoris–Rips complex filtration with distances $0 < \epsilon_1 < \epsilon_2 < \cdots < \epsilon_k$.

```
1: procedure METHODOLOGY
2:     for i = 1 to k do
3:         Draw balls with radius = ε_i and each point in the
       point cloud as center of the ball.
4:         Join all those centers corresponding to which the
       balls intersect. In other words, join all points a_ℓ and
       a_ℓ' if d(a_ℓ, a_ℓ') < 2ε_i.
5:         i = i + 1.
```

[1] If I_j is the jth realization of the index then $I_j(1+\alpha)$ is the jth realization of the index-plus-alpha portfolio.

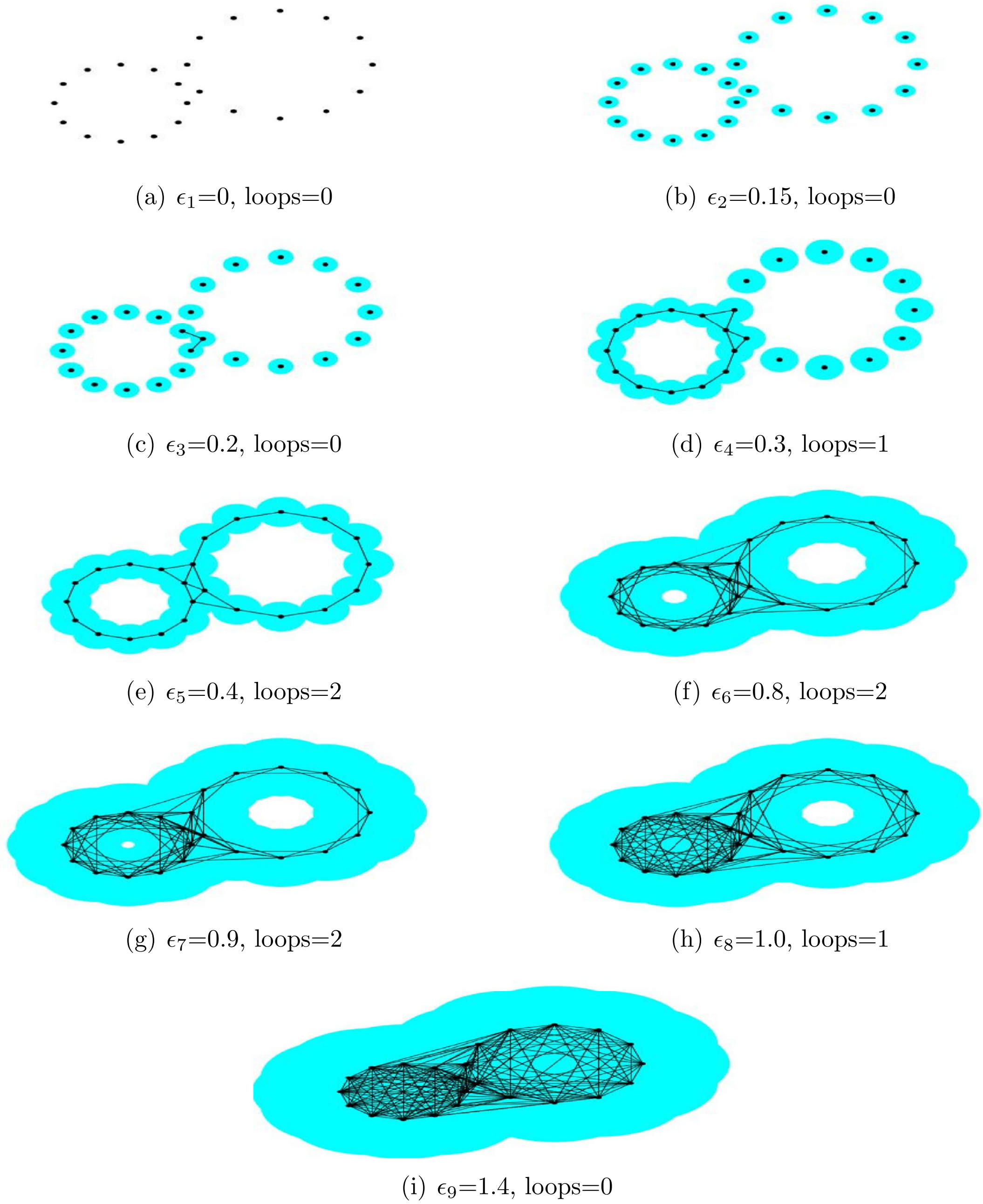


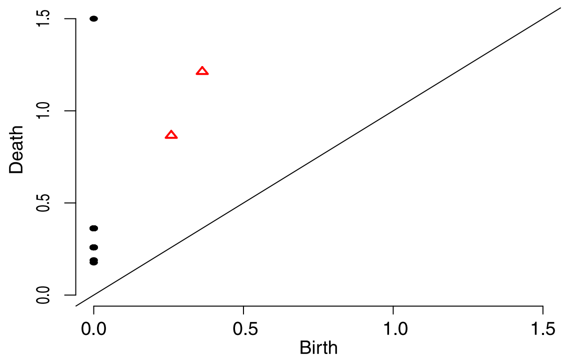
Fig. 1. An illustration to depict a process for generating Rips filtration. Each point in the point cloud is associated with an equal-sized ball; the radius of the ball acts as the filtration parameter. A series of nested simplicial complexes get generated by increasing the value of radius. It results in the appearance and disappearance of features such as connected components and holes. Here, the simplicial complexes illustrate the births and deaths of two holes.

Given the filtration $\mathcal{R} = \{R(P, \epsilon),\ \epsilon > 0\}$, we can compute the k-dimensional homology group $H_k(R(X, \epsilon))$, for each ϵ. The homology changes as ϵ increases, thereby causing the formation of new connected components, loops, and cavities, or destruction of the existing components, loops, and cavities. As an illustration, Fig. 1 depicts the number of loops at each instance.

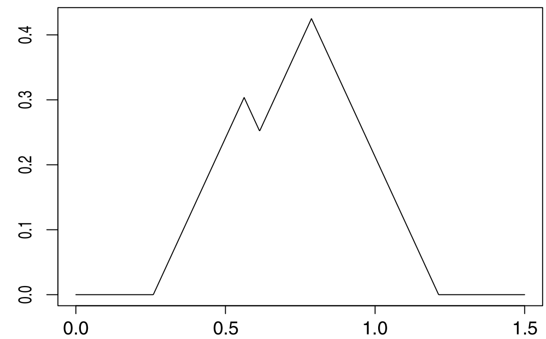
For a fixed k, the lifespan of a k-dimensional component can be described by a point (a, b), where a and b are respectively the birth and death times of that component. The persistence diagram (see, Fig. 2(a)) is the set of all k-dimensional birth-death pairs forming a multiset of points in $\mathbb{R}^2$ (Edelsbrunner & Harer, 2010). More precisely, we analyze the persistence for the j-th topological generator of the k-th dimensional Betti number across different resolution scales through a persistence diagram. In the sequel, we shall be using only the 1-dimensional homology.

2.3. Persistence landscape

Persistent homology allows a multi-scaled view of the data where various qualitative features of the data are tracked across the resolution scales, resulting in a topological signature known as a persistence diagram. Though persistence diagrams contain potentially valuable information about the underlying data set, it is a multi-set which, when equipped with Wasserstein distance, forms an incomplete metric space and hence is not appropriate to apply tools from statistics and machine learning for processing



(a) Persistence diagram



(b) Persistence landscape

Fig. 2. (a) the persistence diagram associated to Fig. 1; the dots in it represent the birth and death of a feature; for instance, two red dots represent the birth and death of two holes in Fig. 1; (b) the corresponding persistence landscape.

topological features. Our aim in this paper is to study time series, and it is useful to embed the persistence diagrams into a Hilbert space to perform the requisite analysis. One such embedding is the persistence landscape, introduced by Bubenik (2015), which is a sequence of real-valued functions that summarize the crucial information contained in persistence diagrams. The persistence landscape representation enjoys several benefits over the persistence diagrams, (Bubenik, 2015; 2018; Chazal et al., 2015; Chazal et al., 2014; Chazal & Michel, 2017) such as they form a Hilbert space and are stable. Moreover, the persistence landscape, by definition, involves no parameter and is thus free from parameter tuning and over-fitting risk (Bubenik, 2015). Another noteworthy aspect of a persistence landscape is that it takes much less time to compute than a persistence diagram (Bubenik, 2018). There are other methods to embed the persistence diagrams such as persistence image, or kernel based methods such as persistence scale-space kernel, to name a few. One can refer to Adams et al. (2017), Carlsson (2009), Gidea and Katz (2018), Reininghaus, Huber, Bauer, and Kwitt (2015) for more description.

To obtain a persistence landscape, firstly, the persistence diagram is rotated clockwise 45°, and then treating the homology feature as a right angle vertex, isosceles right angle triangles are drawn from each feature. From the obtained collection of these triangles, individual functions are obtained. For instance, the third landscape function is the point-wise third maximum of all the triangles drawn. More specifically, with each birth-death pair $p(a, b) \in D$, where D is the persistence diagram, a piece-wise linear function $\Lambda_p : \mathbb{R} \to [0, \infty)$ is associated as follows:

$$\Lambda_p(t) = \begin{cases} t - a & t \in \left[a, \dfrac{a+b}{2}\right] \\ b - t & t \in \left[\dfrac{a+b}{2}, b\right] \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

A persistence landscape (see, Fig. 2(b)) of the birth-death pairs $p_i(a_i, b_i)$, $i = 1, \ldots, m$, is the sequence of functions $\eta : \mathbb{N} \times \mathbb{R} \to [0, \infty)$, as $\eta(k, t) = \eta_k(t)$ where $\eta_k(t)$ denotes the k-th largest value of $\{\Lambda_{p_i}(t),\ i = 1, \ldots, m\}$. We set $\eta_k(x) = 0$ if the k-th largest value does not exist; so, $\eta_k(t) = 0$ for $k > m$.

The persistence landscapes form a subset of the Banach space $L^p(\mathbb{N} \times \mathbb{R})$ consisting of sequences $\eta = (\eta_k)_{k\in\mathbb{N}}$. This set has an obvious vector space structure (Gidea & Katz, 2018), and it becomes a Banach space when endowed with the norm

$$||\eta||_p = \left(\sum_{k=1}^{\infty} ||\eta_k||_p^p\right)^{\frac{1}{p}}, \tag{2}$$

where $||\cdot||_p$ is the L^p-norm. Further, it is shown in Bubenik (2015) that the persistence landscape is stable with respect to the L^p norm for $1 \le p \le \infty$.

2.4. Takens' embedding: state space reconstruction

Another essential ingredient that we shall be using in our research is Takens' embedding theorem. The time delay coordinate embedding or the Takens' embedding is a conventional and most common strategy for state-space reconstruction for nonlinear time series analysis (Horak, Krlín, & Raidl, 2003).

A time series can be treated as a series of projections of the states observed from a dynamical system. The idea of time series analysis is to predict the evolution of the underlying time-varying phenomenon. Generally, it is challenging to reconstruct the state space and the evolution rules of the dynamical system using the time series observations since the distribution of the data generating process is rarely known a priori. Therefore, one typically relies on attractors. An attractor is a set of numerical values toward which the dynamical system tends to evolve for different possible starting states of the system. Since infinitely many points are required to construct any attractor, we instead propose to use time-delay embedding, in particular, Takens' embedding that serves as a quasi-attractor.

Packard, Crutchfield, Farmer, and Shaw (1980) laid the foundation for nonlinear time series analysis, which was carried forward and formalized by Takens (1981). These seminal works introduced the state-space reconstruction allowing one to reconstruct the entire dynamics of a complex nonlinear system from a single time series. The time-delay Takens' embedding has become a useful computational tool to connect a variety of time series with persistent homology by converting the former into a meaningful depiction of point cloud. In doing so, the dynamics reconstructed is the same as that in the original phase space, and hence the topology remains preserved (Takens, 1981).

A time series $x = x_1, x_2, \ldots, x_N$, of a scalar variable x can be reconstructed in phase space in time as follows:

$$X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_{N-(d-1)\tau} \end{bmatrix} = \begin{bmatrix} x_1 & x_{1+\tau} & x_{1+2\tau} \cdots & x_{1+(d-1)\tau} \\ x_2 & x_{2+\tau} & x_{2+2\tau} \cdots & x_{2+(d-1)\tau} \\ \vdots & \vdots & \vdots & \vdots \\ x_{N-(d-1)\tau} & x_{N-(d-2)\tau} & x_{N-(d-3)\tau} \cdots & x_N \end{bmatrix}, \tag{3}$$

Table 1
Data sets considered for empirical analysis in this study.

No.	Index	Country	No. of constituents	Constituents considered
1.	S&P BSE Sensex (Sensex)	India	30	25
2.	CNX Nifty (CNX)	India	50	41
3.	Topix core 30 (Topix)	Japan	30	29
4.	Dow Jones Industrial Average (DJIA)	USA	30	28
5.	DAX 30	Germany	30	27
6.	Athex composite (Athex)	Greece	60	53
7.	S&P Global 100	Global	100	98
8.	IBovespa (Bovespa)	Brazil	60	57
9.	IBX-50 (IBX)	Brazil	50	43
10.	Hang-Seng	Hong Kong	50	46

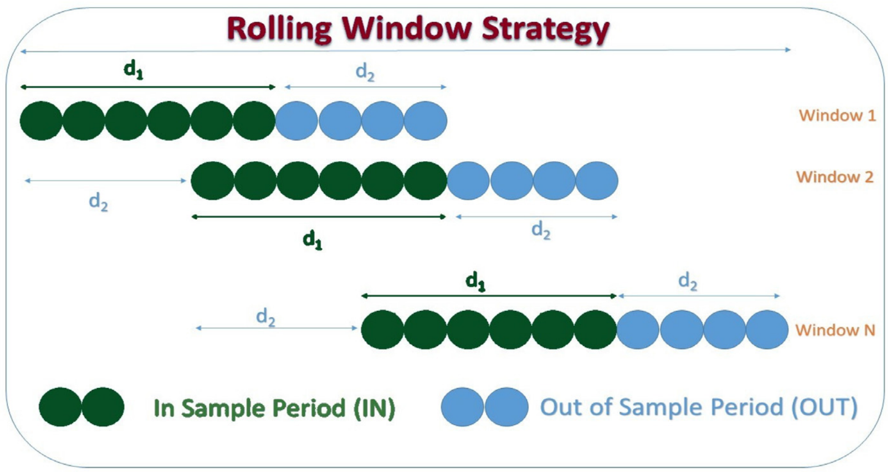


Fig. 3. Illustration of sliding window scheme: here green and blue colors represent in-sample and out-of-sample periods, respectively. For a fixed in-sample period length d_1 and out-of-sample period of length d_2, the first in-sample period (window 1) stands for $1, \ldots, d_1$ with the out-of- sample period $d_1 + 1, \ldots, d_1 + d_2$; the next in-sample period (window 2) gets forward by d_2 to obtain an in-sample window of $1 + d_2, \ldots, d_1 + d_2$, with the corresponding out-of-sample period $d_1 + d_2 + 1, \ldots, d_1 + d_2 + d_2$, and so on.

where τ is the time delay, d is the dimension of reconstructed space (embedding dimension), $N - (d-1)\tau$ is number of points (states) in the phase space, and each point in the space is represented by a row of the matrix X.

Khasawneh and Munch (2014a, 2014b, 2016, 2017) used Takens' embedding, maximal persistence and persistent homology to analyze the stochastic delay equations. More precisely, Khasawneh and Munch (2017) used maximal persistence to analyze Hayes equation and stochastic version Mathieu's equation. Their results indicate that Takens' embedding in combination with TDA is a valid tool for analyzing the stability of stochastic delay equations. Khasawneh and Munch (2014a) applied Takens' embedding to convert the data sets simulated from Euler-Maryuama method to point clouds to study equilibrium and periodic solutions. These studies open the door to apply persistent homology to the analysis of dynamic structures in time series using Takens' embedding. Very recently, Kim, Kim, and Rinaldo (2019) adopted the time-delayed sliding window embedding to develop a novel algorithm for feature extraction in time series data applying tools from TDA.

3. Data and methodology

In this section, we provide details of the data sets and the methodology used in the empirical analysis.

3.1. Sample data

We have used ten data sets in the empirical analysis. The data is extracted from Thompson Reuters EIKON data stream. The period of each index data is taken according to the availability of maximum number of assets in that period. The first 7 data sets consists of the daily closing prices from January 2005 to November 2018 (3625 observations) while the last 3 data sets consist of daily closing prices from January 2010 to November 2018 (2320 observations). Table 1 presents the list of indexes along with their constituents (see, Appendix A for a detailed description of indexes).

3.2. Methodology

The daily returns are calculated using $x_{it} = \ln(\frac{P_{it}}{P_{it-1}})$, $i = 1, \ldots, n, \quad t = 1, \ldots, T$, where P_{it} and P_{it-1} are respectively the closing prices of the i-th asset on t-th and $(t-1)$-th day.

We follow the sliding window approach of in-sample length d_1 and out-of-sample length d_2 on each of the data period considered. That is, the first in-sample period is $1, \ldots, d_1$ with $d_1 + 1, \ldots, d_1 + d_2$ as the out-of-sample period; the next in-sample period gets forward by d_2 to $1 + d_2, \ldots, d_1 + d_2$ with next $d_1 + d_2 + 1, \ldots, d_1 + d_2 + d_2$ out-of-sample period, and so on (see, Fig. 3).

We apply four different pairs of (d_1, d_2), specifically:[2] (126, 63), (126, 42), (126, 21) and (63, 21), in the present analysis.

[2] The values of (d_1, d_2) considered by us are for illustrative purposes. One can easily change them and take some other values for them. However, the values taken in our work are standard in research on investment decision making and portfolio selection. We have considered the in-sample period of 6 months and 3 months following on the lines of Jegadeesh and Titman (2001) and Goel, Sharma, and Mehra (2018b). We varied the out-of-sample period from 1 month, 2 months, and 3 months (Bruni et al., 2017) to obtain a more robust analysis and definite conclusions.

Table 2
Out-of-sample sliding windows in the empirical analysis; d_1 and d_2 represent lengths of in-sample and out-of-sample periods, respectively.

d_1	d_2	Total number of out-of-sample windows	
		data sets 1-7	data set 8-10
126	63	55	34
126	42	83	52
126	21	166	104
63	21	169	107

Table 2 reports the number of out-of-sample periods for each pair (d_1, d_2).

We set the parameters $d = 3$ and $\tau = 1$ in (3)[3].

For the in-sample period of length d_1, and for each asset $i = 1, \ldots, n,$ the one dimensional time series $x_{i1}, x_{i2}, \ldots, x_{id_1},$ of returns of the i-th asset is converted into the multi-dimensional time series using Takens' embedding (3). The resultant matrix of order $((d_1 - 2) \times 3)$ is as follows:

$$\begin{bmatrix} x_{i1} & x_{i2} & x_{i3} \\ x_{i2} & x_{i3} & x_{i4} \\ \vdots & \vdots & \vdots \\ x_{id_1-2} & x_{id_1-1} & x_{id_1} \end{bmatrix}. \tag{4}$$

We choose the time-window of size $\upsilon = 21$ to construct the point cloud formed by an $(\upsilon - (d-1)\tau \times 3$ matrix. For each point cloud, we compute the persistence diagram of Rips filtration, the corresponding persistence landscape, and the L^1-norm (henceforth called the TDA norm). The TDA norms time series is denoted by $\{N_{it}\},\ t = 1, \ldots, d_1 - 22,$ for each asset $i = 1, \ldots, n,$ and in each in-sample period of d_1 days with a sliding step of one day. We apply the same procedure to compute TDA norms $\{\hat{N}_t\}$ of return series of the corresponding market index.

For each in-sample data series of prices, the norms are calculated using Algorithm 2

4. Significance of TDA norm

To illustrate the significance of the TDA norms, we plot the daily return series, standard deviation, and TDA norm series of S&P Global 100 and one of its randomly picked constituent from Jan 2005-No. 2018, in Figs. 4 and 5), respectively.

Comparing the standard deviation with TDA norm in Figs. 4 and 5, we note that the norm values are very high when high volatility persists in returns over an extended period of time, whereas the norm values remain low during periods of low volatility or when high volatility returns lives for a shorter period of time.

In contrast, the two figures clearly distinguish between the usual metric of standard deviation for calculating return volatility and the algebraic structure-based TDA norm metric. The TDA model is robust to outliers and provides a better definition of market signals especially during financial meltdown cycles.

To further confirm our observation, we plot the TDA norms and returns of Athens composite, an index of Greece, which has faced a significant downturn after the onset of the Greek debt crisis from early 2009 till the end of 2018. In Fig. 6, we can observe that the TDA norm values can capture the turbulence prevailing in the market.

Furthermore, for conformity, we calculated the correlation between the TDA norm standard deviation series of S&P Global and Athex Index following sliding window of 252 days in-sample and 126 days out-of-sample period and calculated the average value as well (see, Table 3). No specific trend is observed in correlation of the two and many a times the correlation is less than 0.5, indicating that the TDA norm values are not carrying the same information as the standard deviation.

The above observation motivates us to quantify risk in terms of TDA norm.

[3] There is no generic optimal process to choose the embedding dimension d, and the time delay τ in the transformation process of converting data into a point cloud. It is possible to experiment with several values for (d, τ), though many of them can yield the same topological equivalence (Kim et al., 2019). In the simulation, we use the specific grid values for (d, τ), taking noting from Pereira and de Mello (2015) and Seversky et al. (2016). Michael (2005) (Chapter 1) showed that the best choice for τ is $\tau = 1$ to avoid any possible loss of information from the embedding.

Algorithm 2 A step-wise procedure for creating point clouds

1: **procedure** METHODOLOGY
2: Transform the price series of the in-sample data into log returns using

$$x_{it} \leftarrow \ln \frac{P_{it}}{P_{it-1}}, \quad i = 1, \ldots, n; \quad t = 1, \ldots, d_1,$$

where P_{it} and P_{it-1} are the closing prices of the ith indexon tth and $(t-1)$th day/week, respectively, and d_1 is the number of days in the in-sample period considered.
3: Convert the one dimensional in-sample data of length d_1 into multi-dimensional data using the Takens' embedding theorem to obtain a $((d_1 - (d-1)\tau) \times d)$ matrix. We choose $d = 3$ and $\tau = 1$.

$$\begin{bmatrix} x_{i1} \\ x_{i2} \\ \vdots \\ x_{id_1} \end{bmatrix} \rightarrow \begin{bmatrix} x_{i1} & x_{i2} & x_{i3} \\ x_{i2} & x_{i3} & x_{i4} \\ \vdots & \vdots & \vdots \\ x_{id_1-2} & x_{id_1-1} & x_{id_1} \end{bmatrix} \quad (5).$$

4: The point cloud data set is formed with a time window-size υ and the set is of order $(\upsilon - (d-1)\tau) \times d$. We set $\upsilon = 21$

$$\textit{Point cloud data set 1} \begin{bmatrix} x_{i1} & x_{i2} & x_{i3} \\ x_{i2} & x_{i3} & x_{i4} \\ \vdots & \vdots & \vdots \\ x_{i19} & x_{i20} & x_{i21} \end{bmatrix}.$$

5: A persistence landscape and hence, a TDA norm value is obtained for the point cloud data set.
6: Using a sliding step of one day, other point cloud data set is obtained as

$$\textit{Point cloud data set 2} \begin{bmatrix} x_{i2} & x_{i3} & x_{i4} \\ x_{i3} & x_{i4} & x_{i5} \\ \vdots & \vdots & \vdots \\ x_{i20} & x_{i21} & x_{i22} \end{bmatrix}.$$

7: A persistence landscape and hence, a TDA norm value is obtained for the point cloud data set 2.
8: Following a similar approach, for the matrix of the in-sample period in Step 3, we obtain $d_1 - 22$ point cloud data sets resulting in TDA norm series of length $d_1 - 22$.

5. Application of TDA in enhanced indexing

In this section, we propose to examine if the tools from TDA can be applied to construct a diversified portfolio that outperforms the underlying market index. To do this, we propose a two-step

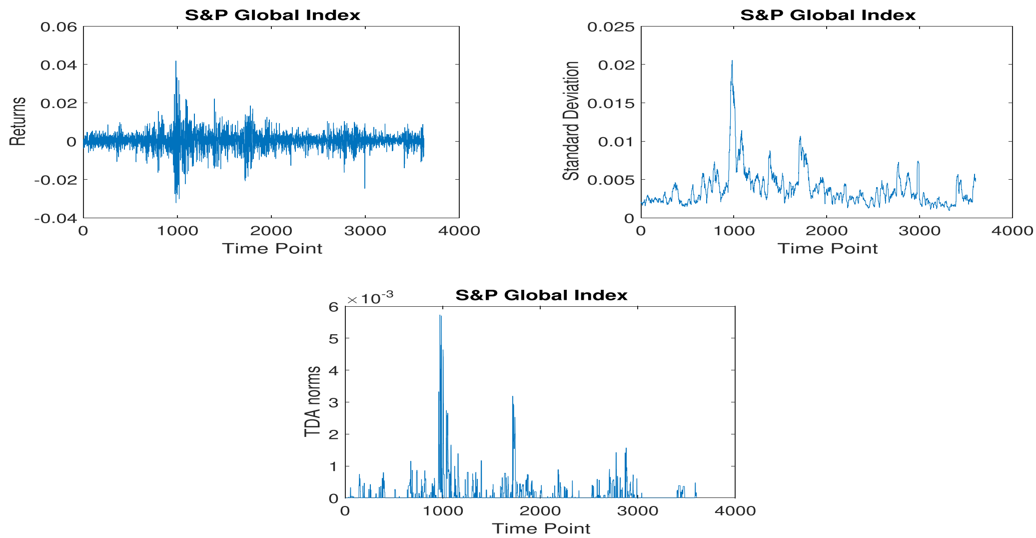


Fig. 4. Top left one is the daily return series, top right one is the corresponding standard deviation series, and the bottom Centre one is the corresponding TDA norm series of S&P Global Index over the period of study.

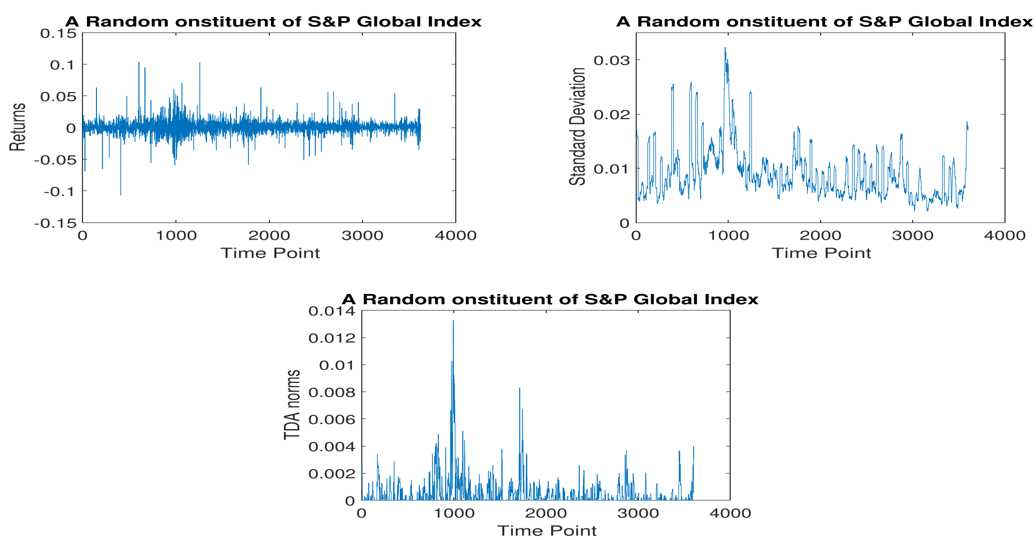


Fig. 5. Top left one is the daily return series, top right one is the corresponding standard deviation series, and the bottom center one is the corresponding TDA norm series of a constituent of S&P Global Index over the period of study.

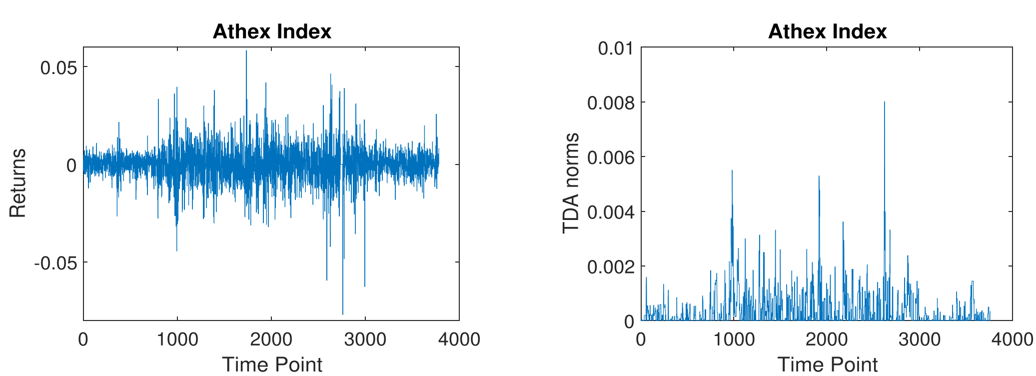


Fig. 6. The daily returns and the corresponding TDA norms of Athens composite index.

Table 3
Correlation between the TDA norm and standard deviation series using sliding window of 252 days in-sample and 126 days out-of-sample period.

Window	S&P Global	Athex
1	0.26	0.08
2	0.13	−0.25
3	0.45	−0.12
4	0.47	−0.04
5	0.57	0.22
6	0.40	0.28
7	0.65	0.72
8	0.55	0.72
9	0.37	0.40
10	0.17	0.32
11	0.15	0.16
12	0.27	−0.01
13	0.48	−0.13
14	0.32	0.11
15	−0.10	0.42
16	0.21	0.41
17	0.21	0.23
18	0.28	0.22
19	0.13	0.25
20	0.39	0.51
21	0.04	0.39
22	0.19	0.13
23	0.26	0.41
24	0.20	0.38
25	0.28	−0.11
26	0.34	0.21
Complete concatenated series	0.55	0.42

process involving filtration of assets and solving an optimization problem. We sketch-out the description in the following.

5.1. Step-1: filtration procedure

We propose Algorithm 3 to divide the whole pool of assets into three categorize based on their TDA norm values.

The idea of creating three bins is adapted from some of the standard approaches in the literature. For instance, the spread between returns from pair of assets is divided by Gatev, Goetzmann, and Rouwenhorst (2006) into three regions to take the trading position in designing pairs trading strategy. Goel et al. (2018b) categorized the assets into three bins based on their potential, measured in the sense of probability, to achieve the goal of the problem statements. Numerous research articles utilized the approach of dividing the stocks into 3 bins on the specific factor as winners (or high-value group), losers (the bottom-most value stocks), and the remaining stocks to form a neutral group, to check the validity of Fama and French factor models in different markets.

To check the adequacy and necessity of the filtering step, we formulate portfolios from bin1 and bin3 applying the Naive strategy[4] and denote them by B1P and B3P, respectively. We also construct a Naive portfolio considering all assets together. The decision for sticking to the Naive strategy instead of solving any risk-return portfolio optimization model is deliberated from some studies (Haley, 2016; 2017) demonstrating that optimal portfolios from mean-risk models struggle to beat the Naive portfolio in the out-of-sample environment. Further, assets in bin2 are susceptible to significant losses on account of their TDA-norm, showing a significant deviation from average norm values. We, therefore, avoided bin2 in portfolio formation.

[4] A Naive portfolio of n assets has $1/n$ weight allocated to each asset.

Algorithm 3 A step-wise procedure for filtering assets in each in-sample period.

1: **procedure** METHODOLOGY

2: Using Algorithm 2, calculate the in-sample TDA norm series of each ith asset constituting the index.

3: For each asset, find the mean value $\bar{N_i}$ of the TDA norm, i. e.

$$\bar{N_i} \leftarrow \frac{1}{d_1-22}\sum_{t=1}^{d_1-22} N_{it}, \quad i = 1, \dots, n.$$

4: For each asset, find the difference df_i of the last TDA norm value from the mean value, i. e.

$$df_i \leftarrow N_{i\,d_1-22} - \bar{N_i}, \quad i = 1, \dots, n.$$

5: Sort the difference values of each assets in ascending order to obtain the sorted series df_{sorted}, a column vector containing n values.

6: Divide the asset class in three categories: bin1, bin2, and bin3.

7: bin1 contains $\lfloor \frac{n}{3} \rfloor$ that correspond to the top $\lfloor \frac{n}{3} \rfloor$ values in df_{sorted}.

8: bin2 contains $\lfloor \frac{n}{3} \rfloor$ assets corresponding to the bottom last $\lfloor \frac{n}{3} \rfloor$ values in df_{sorted}.

9: bin3 contains the remaining $n - 2\lfloor \frac{n}{3} \rfloor$ assets.

5.1.1. Empirical analysis and results

Tables 4–7 in Appendix C present the statistical summary of B1P and B3P. A brief description of performance metrics is available in Appendix B.

Highlighting features of B1P (bin1)

The portfolios attained best Rachev ratio and VaR ratio in all the cases thus maintaining a trade-off between extreme losses and extreme gains. Although risk (measured in terms of VaR, CVaR, Semi-Deviation, MAD, Std Dev.) is moderately high, the portfolios are capable of yielding higher returns thereby acquiring the best Rachev and VaR ratios at both 95% and 97% levels. These portfolios also attain higher mean return values than that from all assets portfolio in 21 out of 40 cases analyzed. The portfolios from bin1 are apposite for investors having moderate to high risk appetite.

Highlighting features of B3P (bin3)

The highlighting finding of these portfolios is their low risk. Risk measured in terms of Std Dev, MAD, Semi-Deviation, Downside deviation, VaR, and CVaR at both 95% and 97% levels are minimum in all the cases. Furthermore, these portfolios have higher Sortino ratio and higher Sharpe ratios in 22 out of 40 cases analyzed. These portfolios are thus suitable for low-risk profile investors.

5.2. Step-2: TDA norm based model for EI

We propose the following EI model that aims to track the index volatility, thereby pushing the beta of portfolio proximate to one. In the same instance, the portfolio must be able to yield returns over and above the index.

$$(ETDA) \quad \min_{w} \sum_{t=1}^{d_1-22} \sum_{i=1}^{n} \left| N_{it} w_i - \hat{N}_t \right|$$
subject to
$$\sum_{i=1}^{n} \mu_i w_i - \mu_I \geq r*,$$
$$\sum_{i=1}^{n} w_i = 1,$$
$$w_i \geq 0, \quad i = 1, \dots, n, \tag{6}$$

where, $\hat{N}_t$ and N_{it} for each $t = 1, \dots, d_1 - 22$, denote the TDA-norms of the benchmark index and asset i, $i = 1, \dots, n$, at the t-th realization, respectively; w_i denotes the proportion invested in the i-th asset with a budget constraint and no short selling restriction; μ_I and $\mu_i = \frac{1}{d_1} \sum_{t=1}^{d_1} x_{it}$, $i = 1, \dots, n$, denote the expected returns from the benchmark index and the i-th asset, in the d_1 days of the in-sample period, respectively; r^* is the desired return over and above the benchmark return from the portfolio.[5]

We use assets in bin1 to construct optimal portfolios by (*ETDA*) model. Our empirical findings thus far suggest that assets in bin1 deliver high risk-adjusted returns. Generally, it is the case that assets with low volatility are not aggressive to yield high growth while high volatility assets often go through sharp ups and downs, and investing in them for a short period could lead to massive losses. A trade-off between enhancing returns and the controlled risk appears by far apter, and hence assets in bin1 are believed to accomplish this feat.

The optimal portfolio formed by assets in bin1 is denoted by *ETDA*1. To perform a comparative analysis, we also construct another optimal portfolio *ETDA*2 from the same assets but by applying model (*ETDA*) minus the constraint (6). The idea is to test if the supplemental constraint on return is adding to the performance of portfolio for EI.

Furthermore, we also considered the following three extensively studied linear models in the fields of EI from the literature. We shall be naming them (*EIM*1), (*EIM*2) and (*EIM*3), respectively.

(*EIM*1): Canakgoz and Beasley (2009) model where the objective is to achieve the beta of the tracking portfolio equals one under the constraint on the alpha of the portfolio, (p. 389, model (34)).

(*EIM*2): Bruni et al. (2017) model based on cumulative zero-order ϵ-stochastic dominance (p. 325, model (12)).

(*EIM*3): Bruni et al. (2015) model which maximizes the excess mean return under an upper bound constraint on the minimax risk measure (p. 739, model (1)).

We use R software with CPLEX solver interface on Widows 64 bits Intel(R) Core(TM) i7-6700 CPU @3.40 GHz processor for solving all optimization problems and employ the R-package TDA and nonlinearTseries which greatly simplifies the application programming interface. The maximum time taken in one in-sample period for the complete process which includes converting one dimensional series to multi-dimensional, computing norms for each of the constituent in the index, filtering assets into bins, solving the proposed model (*ETDA*) and computing the out-of-sample returns, is one minute (approximately).

5.2.1. Empirical analysis and results

Since we have a large number of windows to present the details, we report the outcomes on the series formed by concatenating the out-of-sample returns from each in-sample optimal portfolios. Fig. 7 illustrates the out-of-sample cumulative returns of portfolios from five models on four data sets along with the benchmark index.[6]

Tables 8–11 in Appendix C provide the summary statistics including the mean, minimum value (Min), maximum value (Max), standard deviation (Std Dev), skewness, and kurtosis, for each of the four different in-sample and out-of-sample periods. The tables also summarize the out-of-sample performance of models in terms of seven performance measures namely, excess mean return, Sharpe ratio, Sortino ratio, Omega ratio, Rachev ratio and VaR ratio (at both 95% and 97% levels). The best values among the five models are highlighted in bold.

Empirical results

1. Excess Mean Return (EMR): The *ETDA*1 and *ETDA*2 portfolios achieve the highest EMR for 13 and 16 cases, respectively, out of 40 cases amongst all models. These two models provides better returns over and above the index.
2. Risk-reward ratios: These ratios reflect the overall performance of a given strategy by combining the values of mean return and risk and thus are more relevant in comparative analysis.

    Sharpe Ratio and Sortino Ratio: The standard deviation is relatively high in case of *ETDA*1 and *ETDA*2 portfolios, and hence these two experience higher values of Sharpe/Sortino ratios in 9/8 and 10/11 times only, out of 40.

    Rachev ratio at $\alpha_1 = \alpha_2 = 95\%,\ 97\%$: Here, *ETDA*1 and *ETDA*2 achieve the highest values in all 40 cases. The silver lining is that these are the only portfolios achieving first and second highest values.

    VaR ratio at $\alpha_1 = \alpha_2 = 95\%,\ 97\%$: With highest values for 27 cases and the second highest for the remaining, *ETDA*1 excels in this performance measure. Similarly, *ETDA*2 yields the highest values in 13 and second highest in the remaining cases. The *ETDA*1 and *ETDA*) are the only players in this front beating the rest three.

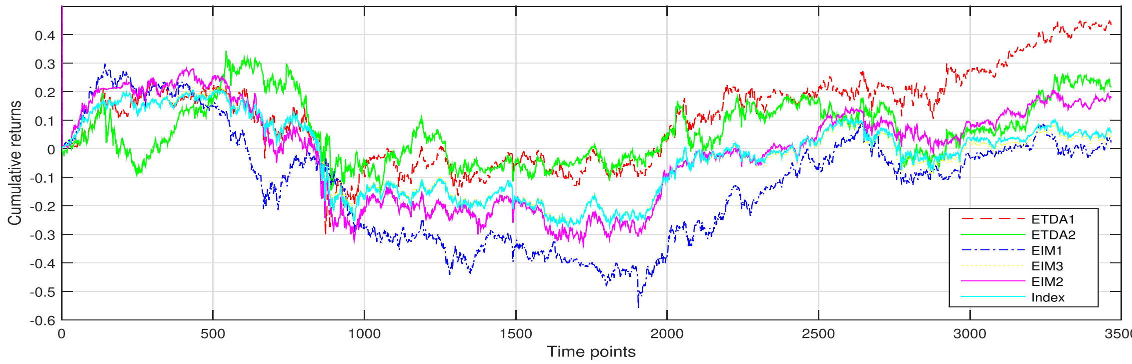
We observe that both (*ETDA*) and (*ETDA*) without constraint on the return are not the best models among the five on the Sharpe ratio. The value of the Sharpe ratio depends on the standard deviation, which takes the high value with high volatility. But, in the world of finance, volatility is not considered to be an enemy. It is only the downside potential that causes worry. An investment could have a low Sharpe ratio and still provide excellent risk-adjusted returns if the cause of its high standard deviation is due to upside potential only.

On the other hand VaR and Rachev ratios capturing the tail loses are crucial factors in EI. Our analysis indicate that (*ETDA*) and (*ETDA*) without constraint on the return excel on these two ratios at different confidence levels in all cases and on all data sets. We can conclude that the proposed models are strong competitors to some of the existing conventional models for EI.

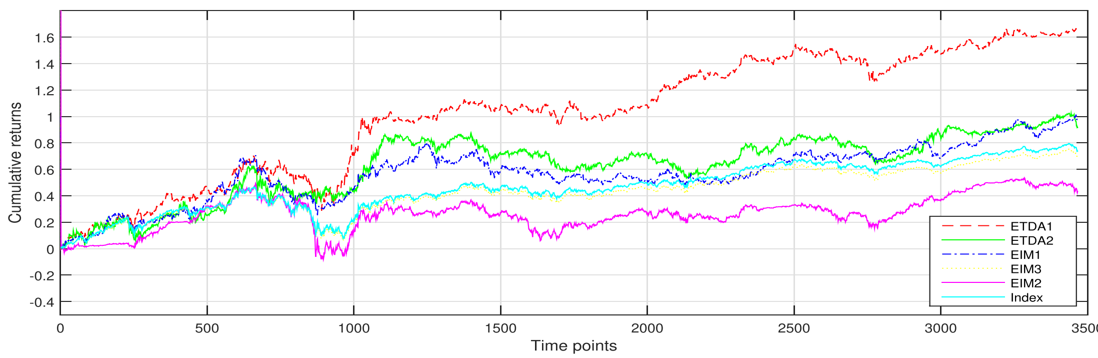
The possible reason for the observed improvement in the reward-risk ratios capturing tails is the observation in Section 4 that the TDA norm is proficient in capturing the bearish phase, and the value of the TDA norm shoots up drastically if stress in the market persists for long. The TDA norm thus measures the sharp downward trends from the index very well compared to the standard deviation representing both the upward and downward deviations from the index. This property of the TDA norm prevents the portfolio from suffering extreme losses and hence hedge against massive losses. Moreover, our analysis

[5] In the present analysis, we take $r* = 2\%$ per annum.

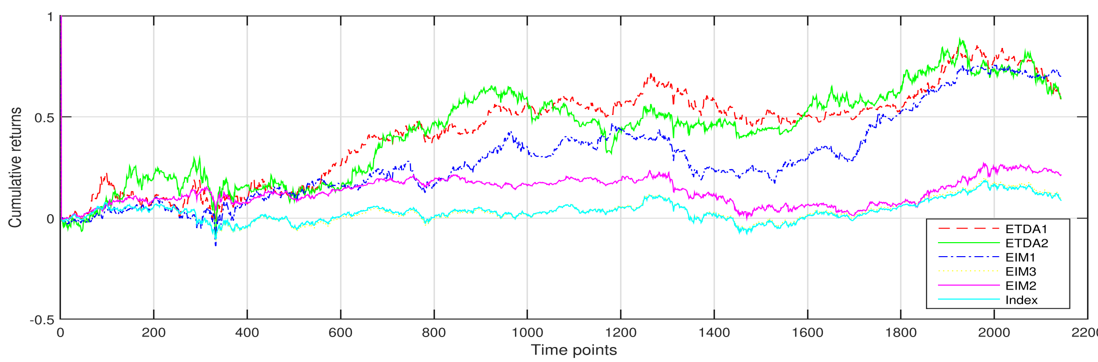
[6] Four indices are selected for illustrative purposes only.



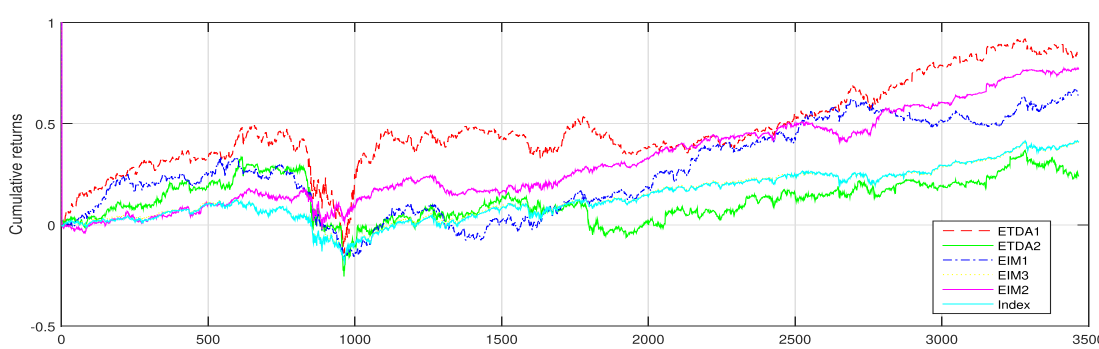
(a) Topix Index



(b) Sensex 30 Index



(c) Hang seng Index



(d) Dow Jones Index

Fig. 7. Out-of-sample cumulative returns of optimal portfolios from EI models.

indicates that TDA norm is not just the standard deviation or pure volatility but carries different information. However, it needs further exploration of what equivalence it has with other risk measures, especially lower tail risk.

6. Conclusions

We investigated the use of topological data analysis on the one-dimensional time series of returns of assets to uncover some properties in the financial data. Using Takens' embedding, we converted the time series to a point cloud representing the states of its dynamical system. We employed the L^p-norm of the persistence landscapes as a quantifier of stability of topological features; the higher the norm value, the more is the instability, and hence the volatility is high. We propose a two-step application of TDA in finance in enhanced indexing.

We proposed a new optimization model for enhanced indexing by minimizing the tracking error defined as the absolute difference in the portfolio TDA-norm and the tracked index TDA-norm. In the proposed two-step procedure, we first filter out assets into three bins and select the one having the best performance in terms of reward-risk ratios to meet the objective of enhanced indexing.

Using the sliding window approach, and applying the Naive strategy, we constructed the portfolios from each of the two bins on the 10 data sets. Our empirical analysis of various financial performance indicators indicated that the bin1 portfolios possess higher values of the reward-risk ratios, and hence felicitous for investors with a remotely high-risk appetite. In the next step, we obtain the optimal weights by solving the proposed model on the filtered assets only.

The comparison of the proposed model with the several existing models for enhanced indexing is performed using the sliding window analysis on 10 data sets covering an extended period. The proposed model is found to have a higher excess mean return, risk-reward ratios such as VaR ratio, and Rachev ratio, than the other models.

In this article, the embedding dimension, d is set 3. In the future, we plan to find the optimal embedding length for financial data sets. Moreover, we observed from our empirical analysis that the TDA norm is not just the standard deviation or volatility but carries more vital information, which resulted in better performance of the portfolios. It is a crucial point for exploring the relationship between TDA norm and other risk measures, especially lower tail risk measures, which could lead to interesting and useful results in investment decision problems.

Although TDA accommodates far more rich geometry and structural information about the data, it is not yet entirely explored to its potential in investment decisions, including portfolio optimization. This paper has been an endeavor to decipher the application of TDA in asset allocation. The superior practical performance of the proposed models designates the promise that the TDA holds in problems of finance and econometrics. However, researchers should be critical about these methodologies as they are relatively new applications to the field of finance.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Credit authorship contribution statement

Anubha Goel: Conceptualization, Data curation, Formal analysis, Investigation, Methodology, Visualization, Writing - original draft, Writing - review & editing. **Puneet Pasricha:** Investigation, Visualization, Writing - review & editing. **Aparna Mehra:** Conceptualization, Project administration, Resources, Software, Supervision, Writing - original draft, Writing - review & editing.

Acknowledgments

The authors are very thankful to the esteemed reviewers for their insightful and constructive commendations that led to an improved version of the current article. The authors acknowledge the positive support from the editor-in-chief and associate editor. The authors are grateful to Professor Damir Filipovic, EPFL Switzerland, for a quick read of the paper and providing feedback on personal request. The first author is also thankful to the Council of Scientific and Industrial Research (CSIR), India, for the financial grant (no. 09/086(1215)/2015-EMR-I), for initiating this research and subsequently Indian Institute of Technology Delhi, India, for extending grant during the period of revision process.

Appendix A

The detailed description of the indices considered in the study is as follows:

1. S&P BSE Sensex (Sensex): It consists of 30 largest and most liquid stocks listed under Bombay Stock Exchange (BSE), the oldest stock exchange in Asia. The data consists of 25 assets.
2. CNX Nifty (CNX): The key index of 50 major stocks listed in the National Stock Exchange (NSE), India. The data comprises of 41 assets.
3. Topix core 30 (Topix): The Topix Core 30 is an index of the 30 most liquid and highly capitalized stocks of the over 1500 companies listed on the Topix index or Japan's Tokyo Stock Exchange. The data consists of 29 assets.
4. Dow Jones Industrial Average (DJIA): The DJIA is a US stock market index. The data has 28 assets.
5. DAX 30: The DAX is a bluechip stock market index consisting of 30 major German companies trading on the Frankfurt Stock Exchange. The data consists of 27 constituents.
6. Athex composite (Athex): It is listed on Athens exchange and consists of 60 constituents. The data contains 53 assets.
7. S&P Global 100: This index measures the performance of bluechip companies of major importance in the global equity markets. The data comprises of 98 assets.
8. IBovespa (Bovespa): It is the Brazilian benchmark index of about 60 stocks. The data includes 57 assets.
9. IBX-50 (IBX): The Brazil 50 index tracking 50 key stocks listed on the stock exchange of Sao Paulo, Brazil. The data contains 43 assets.
10. Hang-Seng: This index is a free float-adjusted market-capitalization-weighted stock-market index in Hong Kong consisting of 50 companies. The data embodies 46 constituents.

Appendix B

We briefly explain the performance measures considered in the out-of-sample analysis in the paper. For more details, we suggest to refer to Bacon (2011).

Excess Mean Return (EMR)

It is an expected value of the difference between portfolio return and index return, that is, $EMR = E(w) - E(I)$, where $E(w)$ and $E(I)$ denote the mean returns from the portfolio $w = (w_1, \ldots, w_n)'$, and the benchmark index, respectively. Higher values of EMR are desired.

Value-at-Risk (VaR_α)

It measures the maximum loss in portfolio at a confidence level $\alpha \in (0, 1)$, and it is defined as:

$$VaR_\alpha(-w) = \min\{r \in \mathbb{R} \mid F_{-w}(r) = Pr(-w \leq r) \geq \alpha\},$$

where $F_{-w}(r)$ is the distribution for the portfolio loss $-w$.

Condition value at Risk (CVaR_α)

It is the mean of losses in portfolio beyond $VaR_\alpha(-w)$.

Sharpe Ratio (SR)

It is the average return earned in excess of the risk-free rate per unit of volatility as measured by the standard deviation and is defined as:

$$\text{SR} = \begin{cases} \dfrac{E(w) - r_f}{\sigma(w)} & E(w) > r_f \\ 0 & E(w) \leq r_f. \end{cases} \tag{7}$$

where r_f is the risk-free rate and $\sigma(w)$ is the standard deviation of portfolio returns. If $\sigma(w)$ in (7) is replaced by the modified VaR or CVaR then the resulting ratio is called the "modified Sharpe ratio".

Sortino Ratio

It is the ratio of mean return to the risk measured by under achievement of portfolio from the benchmark or the risk free return. It is defined by

$$\text{Sortino ratio} = \begin{cases} \dfrac{E(w) - r_f}{\sqrt{\sum_{t=1}^{T} \dfrac{[-(-x_t + r_f)^+]^2}{T}}} & E(w) > r_f \\ 0 & E(w) \leq r_f. \end{cases}$$

where x_t is the tth realization, $t = 1, \ldots, T$, of portfolio w, and r_f is the risk free return, and $\xi^+ = \max\{0, \xi\}$.

Downside Deviation (DD)

It depicts under achievement of portfolio from the benchmark or the risk free return. It is given by

$$\text{DD} = \sqrt{\sum_{t=1}^{T} \frac{((r_f - x_t)^+)^2}{T}}.$$

The lower values are preferable.

Rachev Ratio (RR)

It is defined as follows:

$$RR_{\alpha_1,\alpha_2} = \frac{CVaR_{\alpha_1}(w)}{CVaR_{\alpha_2}(-w)}, \quad \alpha_1, \alpha_2 \in (0, 1).$$

In case of enhanced indexing, we replace (w) by $(w - I)$, where I is the benchmark index. Intuitively, this ratio allows a trade-off between potential of extreme positive returns to the risk of extreme losses.

VaR Ratio (VR)

It is the same as Rachev ratio with VaR measure replacing the CVaR in the ratio.

Note: In our experiments, the risk-free rate is considered to be zero.

Appendix C

The part contains out-of-sample statistics on ten data sets from our experiments.

Table 4
Out-of-sample performance analysis of the filtered portfolios using the concatenated series with 6 months and 3 months in-sample and out-of-sample periods, respectively. In case the mean return is negative, the values of Sharpe ratio and Sortino ratios are taken to be zero.

6–3	Topix			S&P Global			DAX 30			Dow Jones			Hang-Seng		
	B1P	B3P	All	B1P	B3P	All	B1P	B3P	All	B1P	B3P	All	B1P	B3P	All
Mean	6.760E−05	6.560E−05	**7.590E−05**	5.100E−05	**6.570E−05**	6.310E−05	**1.181E−04**	4.980E−05	1.034E−04	1.244E−04	**1.609E−04**	1.386E−04	**1.314E−04**	9.660E−05	1.136E−04
Min	−4.706E−02	−4.447E−02	−4.407E−02	−4.055E−02	−3.118E−02	−3.301E−02	−4.184E−02	−3.351E−02	−3.729E−02	−4.483E−02	−3.288E−02	−3.810E−02	−2.755E−02	−2.398E−02	−2.472E−02
Max	5.662E−02	5.406E−02	5.199E−02	4.494E−02	3.934E−02	4.459E−02	4.843E−02	4.282E−02	5.086E−02	4.942E−02	4.772E−02	5.143E−02	2.759E−02	2.700E−02	2.693E−02
Std Dev	6.566E−03	**5.819E−03**	5.956E−03	5.200E−03	**4.047E−03**	4.581E−03	6.161E−03	**5.291E−03**	5.599E−03	5.581E−03	**4.369E−03**	4.797E−03	5.299E−03	**4.327E−03**	4.658E−03
MAD	4.529E−03	**3.887E−03**	4.011E−03	3.469E−03	**2.690E−03**	2.998E−03	4.282E−03	**3.648E−03**	3.781E−03	3.523E−03	**2.858E−03**	2.978E−03	3.828E−03	**3.060E−03**	3.318E−03
Semi Dev	4.751E−03	**4.256E−03**	4.361E−03	3.823E−03	**2.992E−03**	3.372E−03	4.564E−03	**3.906E−03**	4.140E−03	4.053E−03	**3.129E−03**	3.473E−03	3.838E−03	**3.148E−03**	3.402E−03
DD	4.719E−03	**4.226E−03**	4.326E−03	3.800E−03	**2.963E−03**	3.344E−03	4.509E−03	**3.883E−03**	4.093E−03	4.000E−03	**3.056E−03**	3.414E−03	3.773E−03	**3.101E−03**	3.348E−03
Sortino ratio	1.433E−02	1.552E−02	**1.755E−02**	1.341E−02	**2.218E−02**	1.887E−02	**2.619E−02**	1.282E−02	2.526E−02	3.111E−02	**5.264E−02**	4.058E−02	**3.482E−02**	3.114E−02	3.393E−02
$SR_{Std.Dev}$	1.030E−02	1.127E−02	**1.275E−02**	9.802E−03	**1.623E−02**	1.377E−02	**1.917E−02**	9.406E−03	1.847E−02	2.230E−02	**3.683E−02**	2.889E−02	**2.479E−02**	2.232E−02	2.439E−02
SR_{CVaR}	4.348E−03	4.602E−03	**5.206E−03**	3.902E−03	**6.499E−03**	5.483E−03	**7.794E−03**	3.784E−03	7.396E−03	8.971E−03	**1.537E−02**	1.170E−02	**1.063E−02**	9.361E−03	1.021E−02
SR_{VaR}	6.689E−03	7.393E−03	**8.385E−03**	6.233E−03	**1.055E−02**	9.179E−03	**1.180E−02**	5.865E−03	1.117E−02	1.557E−02	**2.455E−02**	1.936E−02	**1.518E−02**	1.388E−02	**1.518E−02**
$CVaR_{97\%}$	1.849E−02	**1.715E−02**	1.758E−02	1.572E−02	**1.231E−02**	1.400E−02	1.794E−02	**1.565E−02**	1.658E−02	1.701E−02	**1.258E−02**	1.440E−02	1.434E−02	**1.205E−02**	1.298E−02
$CVaR_{95\%}$	1.556E−02	**1.425E−02**	1.458E−02	1.306E−02	**1.011E−02**	1.151E−02	1.515E−02	**1.315E−02**	1.398E−02	1.387E−02	**1.047E−02**	1.184E−02	1.236E−02	**1.032E−02**	1.112E−02
$VaR_{97\%}$	1.270E−02	**1.114E−02**	1.153E−02	1.019E−02	**7.798E−03**	9.027E−03	1.212E−02	**1.059E−02**	1.147E−02	1.069E−02	**8.252E−03**	9.052E−03	1.027E−02	**8.629E−03**	9.072E−03
$VaR_{95\%}$	1.011E−02	**8.870E−03**	9.054E−03	8.178E−03	**6.227E−03**	6.875E−03	1.001E−02	**8.486E−03**	9.256E−03	7.992E−03	**6.554E−03**	7.157E−03	8.653E−03	**6.960E−03**	7.483E−03
$RR_{97\%,97\%}$	**3.097E−04**	2.579E−04	2.700E−04	**2.136E−04**	1.310E−04	1.703E−04	**2.753E−04**	2.081E−04	2.362E−04	**2.665E−04**	1.502E−04	1.942E−04	**1.896E−04**	1.315E−04	1.514E−04
$RR_{95\%,95\%}$	**2.255E−04**	1.813E−04	1.907E−04	**1.512E−04**	8.990E−05	1.165E−04	**1.995E−04**	1.492E−04	1.680E−04	**1.792E−04**	1.038E−04	1.301E−04	**1.430E−04**	9.710E−05	1.123E−04
$VR_{97\%,97\%}$	**1.550E−04**	1.114E−04	1.230E−04	**9.580E−05**	5.670E−05	7.170E−05	**1.311E−04**	9.740E−05	1.114E−04	**1.055E−04**	6.340E−05	7.390E−05	**1.009E−04**	6.730E−05	7.680E−05
$VR_{95\%,95\%}$	**1.035E−04**	7.600E−05	7.880E−05	**6.310E−05**	3.550E−05	4.400E−05	**9.150E−05**	6.580E−05	7.310E−05	**6.300E−05**	4.060E−05	4.670E−05	**7.150E−05**	4.640E−05	5.400E−05
	Sensex 30			CNX			IBX-50			Bovespa			Athex		
Mean	2.445E−04	**2.611E−04**	2.451E−04	2.409E−04	**2.562E−04**	2.545E−04	1.580E−05	**1.654E−04**	6.740E−05	2.410E−05	**1.232E−04**	5.560E−05	−1.459E−04	−1.460E−04	**−7.310E−05**
Min	−5.297E−02	−3.974E−02	−4.804E−02	−4.891E−02	−3.928E−02	−4.459E−02	−4.342E−02	−4.331E−02	−4.264E−02	−4.703E−02	−3.873E−02	−4.315E−02	−9.386E−02	−7.322E−02	−6.725E−02
Max	7.368E−02	5.589E−02	6.632E−02	6.415E−02	5.580E−02	6.009E−02	3.006E−02	4.318E−02	2.545E−02	2.976E−02	3.468E−02	2.455E−02	5.713E−02	**7.124E−02**	5.214E−02
Std Dev	6.657E−03	**5.622E−03**	5.863E−03	6.256E−03	**5.274E−03**	5.591E−03	6.620E−03	**5.380E−03**	5.626E−03	6.297E−03	**4.923E−03**	5.438E−03	8.996E−03	7.759E−03	**6.896E−03**
MAD	4.616E−03	**3.865E−03**	3.956E−03	4.358E−03	**3.603E−03**	3.784E−03	4.837E−03	**3.866E−03**	4.118E−03	4.597E−03	**3.591E−03**	3.960E−03	5.997E−03	5.242E−03	**4.641E−03**
Semi Dev	4.770E−03	**4.053E−03**	4.221E−03	4.512E−03	**3.879E−03**	4.098E−03	4.697E−03	**3.798E−03**	4.028E−03	4.495E−03	**3.507E−03**	3.919E−03	6.648E−03	5.724E−03	**5.137E−03**
DD	4.653E−03	**3.931E−03**	4.108E−03	4.397E−03	**3.762E−03**	3.983E−03	4.689E−03	**3.715E−03**	3.994E−03	4.483E−03	**3.444E−03**	3.891E−03	6.714E−03	5.791E−03	**5.171E−03**
Sortino ratio	5.255E−02	**6.644E−02**	5.965E−02	5.479E−02	**6.810E−02**	6.390E−02	3.380E−03	**4.454E−02**	1.687E−02	5.384E−03	**3.576E−02**	1.429E−02	–	–	–
$SR_{Std.Dev}$	3.674E−02	**4.645E−02**	4.180E−02	3.851E−02	**4.858E−02**	4.551E−02	2.394E−03	**3.075E−02**	1.198E−02	3.832E−03	**2.502E−02**	1.023E−02	–	–	–
SR_{CVaR}	1.568E−02	**1.960E−02**	1.738E−02	1.624E−02	**2.000E−02**	1.859E−02	1.073E−03	**1.404E−02**	5.328E−03	1.718E−03	**1.138E−02**	4.523E−03	–	–	–
SR_{VaR}	2.428E−02	**3.015E−02**	2.731E−02	2.500E−02	**3.292E−02**	2.983E−02	1.475E−03	**2.026E−02**	7.650E−03	2.336E−03	**1.620E−02**	6.399E−03	–	–	–
$CVaR_{97\%}$	1.853E−02	**1.574E−02**	1.676E−02	1.755E−02	**1.540E−02**	1.637E−02	1.675E−02	**1.366E−02**	1.456E−02	1.596E−02	**1.253E−02**	1.414E−02	2.704E−02	2.272E−02	**2.032E−02**
$CVaR_{95\%}$	1.560E−02	**1.332E−02**	1.410E−02	1.484E−02	**1.281E−02**	1.369E−02	1.477E−02	**1.178E−02**	1.265E−02	1.405E−02	**1.082E−02**	1.229E−02	2.237E−02	1.927E−02	**1.722E−02**
$VaR_{97\%}$	1.274E−02	**1.101E−02**	1.134E−02	1.226E−02	**1.027E−02**	1.110E−02	1.294E−02	**9.933E−03**	1.097E−02	1.198E−02	**8.974E−03**	1.060E−02	1.755E−02	1.587E−02	**1.375E−02**
$VaR_{95\%}$	1.007E−02	**8.662E−03**	8.975E−03	9.636E−03	**7.783E−03**	8.531E−03	1.074E−02	**8.164E−03**	8.809E−03	1.033E−02	**7.605E−03**	8.690E−03	1.385E−02	1.273E−02	**1.143E−02**
$RR_{97\%,97\%}$	**3.239E−04**	2.373E−04	2.672E−04	**2.863E−04**	2.081E−04	2.376E−04	**2.843E−04**	1.928E−04	2.047E−04	**2.538E−04**	1.556E−04	1.900E−04	**6.428E−04**	4.536E−04	3.617E−04
$RR_{95\%,95\%}$	**2.334E−04**	1.717E−04	1.897E−04	**2.065E−04**	1.482E−04	1.681E−04	**2.177E−04**	1.437E−04	1.561E−04	**1.946E−04**	1.174E−04	1.456E−04	**4.454E−04**	3.213E−04	2.591E−04
$VR_{97\%,97\%}$	**1.588E−04**	1.176E−04	1.217E−04	**1.396E−04**	9.900E−05	1.088E−04	**1.660E−04**	1.040E−04	1.150E−04	**1.391E−04**	8.240E−05	1.087E−04	**2.765E−04**	2.064E−04	1.659E−04
$VR_{95\%,95\%}$	**1.026E−04**	7.560E−05	7.980E−05	**9.120E−05**	6.150E−05	6.980E−05	**1.129E−04**	7.060E−05	7.970E−05	**1.036E−04**	6.000E−05	7.650E−05	**1.756E−04**	1.380E−04	1.111E−04

Table 5
Out-of-sample performance analysis of the filtered portfolios using the concatenated series with 6 months and 2 months In-sample and Out-of-sample periods respectively.

6–2	Topix			S&P Global			DAX 30			Dow Jones			Hang-seng		
	B1P	B3P	All	B1P	B3P	All	B1P	B3P	All	B1P	B3P	All	B1P	B3P	All
Mean	**7.010E−05**	6.780E−05	6.770E−05	**2.972E−04**	2.531E−04	2.580E−04	1.363E−04	**1.732E−04**	1.376E−04	2.573E−04	**2.689E−04**	2.471E−04	**1.066E−04**	7.520E−05	1.053E−04
Min	−5.076E−02	**−4.343E−02**	−4.407E−02	−5.235E−02	**−3.787E−02**	−4.459E−02	−3.854E−02	−4.011E−02	**−3.810E−02**	−5.932E−02	**−3.982E−02**	−4.804E−02	−2.998E−02	**−2.244E−02**	−2.472E−02
Max	**6.083E−02**	4.438E−02	5.199E−02	**6.415E−02**	5.580E−02	6.009E−02	4.984E−02	4.420E−02	**5.143E−02**	7.368E−02	5.589E−02	**6.632E−02**	**3.071E−02**	1.920E−02	2.693E−02
Std Dev	6.812E−03	**5.774E−03**	5.961E−03	6.554E−03	**5.262E−03**	5.592E−03	5.549E−03	**4.479E−03**	4.805E−03	6.996E−03	**5.617E−03**	5.861E−03	5.578E−03	**4.275E−03**	4.706E−03
MAD	4.600E−03	**3.893E−03**	4.018E−03	4.480E−03	**3.609E−03**	3.790E−03	3.531E−03	**2.914E−03**	2.990E−03	4.706E−03	**3.871E−03**	3.960E−03	3.999E−03	**3.070E−03**	3.349E−03
Semi Dev	4.947E−03	**4.227E−03**	4.366E−03	4.741E−03	**3.847E−03**	4.095E−03	3.992E−03	**3.238E−03**	3.481E−03	5.039E−03	**4.035E−03**	4.218E−03	4.023E−03	**3.139E−03**	3.435E−03
DD	4.914E−03	**4.196E−03**	4.335E−03	4.603E−03	**3.731E−03**	3.978E−03	3.933E−03	**3.162E−03**	3.422E−03	4.921E−03	**3.909E−03**	4.104E−03	3.970E−03	**3.103E−03**	3.384E−03
Sortino ratio	1.427E−02	**1.617E−02**	1.561E−02	6.456E−02	**6.783E−02**	6.486E−02	3.465E−02	**5.477E−02**	4.021E−02	5.229E−02	**6.879E−02**	6.021E−02	2.684E−02	2.423E−02	**3.113E−02**
$SR_{Std.Dev}$	1.030E−02	**1.175E−02**	1.135E−02	4.535E−02	**4.809E−02**	4.614E−02	2.456E−02	**3.866E−02**	2.864E−02	3.678E−02	**4.787E−02**	4.216E−02	1.911E−02	1.759E−02	**2.239E−02**
SR_{CVaR}	4.276E−03	**4.767E−03**	4.631E−03	1.891E−02	**1.976E−02**	1.888E−02	9.987E−03	**1.603E−02**	1.158E−02	1.533E−02	**2.022E−02**	1.756E−02	8.137E−03	7.356E−03	**9.374E−03**
SR_{VaR}	6.961E−03	**7.709E−03**	7.447E−03	2.998E−02	**3.067E−02**	3.035E−02	1.714E−02	**2.547E−02**	1.902E−02	2.445E−02	**3.079E−02**	2.753E−02	1.176E−02	1.074E−02	**1.370E−02**
$CVaR_{97\%}$	1.971E−02	**1.709E−02**	1.755E−02	1.870E−02	**1.524E−02**	1.632E−02	1.660E−02	**1.298E−02**	1.441E−02	2.016E−02	**1.561E−02**	1.671E−02	1.517E−02	**1.186E−02**	1.308E−02
$CVaR_{95\%}$	1.640E−02	**1.423E−02**	1.461E−02	1.572E−02	**1.281E−02**	1.366E−02	1.365E−02	**1.080E−02**	1.188E−02	1.679E−02	**1.330E−02**	1.407E−02	1.310E−02	**1.022E−02**	1.124E−02
$VaR_{97\%}$	1.283E−02	**1.132E−02**	1.163E−02	1.293E−02	**1.036E−02**	1.108E−02	1.063E−02	**8.421E−03**	9.104E−03	1.347E−02	**1.098E−02**	1.130E−02	1.139E−02	**8.730E−03**	9.379E−03
$VaR_{95\%}$	1.007E−02	**8.801E−03**	9.086E−03	9.913E−03	**8.252E−03**	8.500E−03	7.951E−03	**6.798E−03**	7.236E−03	1.052E−02	**8.732E−03**	8.975E−03	9.060E−03	**6.999E−03**	7.692E−03
$RR_{97\%,97\%}$	**3.538E−04**	2.564E−04	2.689E−04	**3.264E−04**	2.055E−04	2.363E−04	**2.593E−04**	1.582E−04	1.937E−04	**3.814E−04**	2.345E−04	2.655E−04	**2.211E−04**	1.222E−04	1.545E−04
$RR_{95\%,95\%}$	**2.501E−04**	1.805E−04	1.907E−04	**2.323E−04**	1.478E−04	1.678E−04	**1.755E−04**	1.091E−04	1.306E−04	**2.664E−04**	1.713E−04	1.892E−04	**1.627E−04**	9.310E−05	1.148E−04
$VR_{97\%,97\%}$	**1.567E−04**	1.153E−04	1.238E−04	**1.536E−04**	9.770E−05	1.087E−04	**1.017E−04**	6.560E−05	7.470E−05	**1.668E−04**	1.166E−04	1.210E−04	**1.170E−04**	6.870E−05	8.050E−05
$VR_{95\%,95\%}$	**1.015E−04**	7.500E−05	7.900E−05	**9.770E−05**	6.580E−05	7.000E−05	**6.270E−05**	4.330E−05	4.750E−05	**1.097E−04**	7.760E−05	8.080E−05	**7.460E−05**	4.790E−05	5.610E−05
	Sensex 30			CNX			IBX−50			Bovespa			Athex		
Mean	**1.226E−04**	4.910E−05	9.880E−05	**5.950E−05**	3.460E−05	6.050E−05	3.540E−05	**1.276E−04**	9.010E−05	5.160E−05	**1.060E−04**	8.260E−05	−8.369E−05	−1.527E−04	**−7.310E−05**
Min	−4.665E−02	**−3.552E−02**	−3.729E−02	−3.364E−02	**−2.980E−02**	−3.301E−02	−4.342E−02	−4.331E−02	**−4.264E−02**	−4.703E−02	**−3.873E−02**	−4.315E−02	−9.091E−02	−6.609E−02	−6.725E−02
Max	**5.900E−02**	4.903E−02	5.086E−02	4.255E−02	**4.467E−02**	4.459E−02	**3.317E−02**	2.177E−02	2.545E−02	3.213E−02	**3.354E−02**	2.455E−02	**6.624E−02**	5.054E−02	5.214E−02
Std Dev	6.498E−03	**5.211E−03**	5.596E−03	5.406E−03	**4.094E−03**	4.578E−03	6.775E−03	**5.058E−03**	5.670E−03	6.507E−03	**4.857E−03**	5.490E−03	8.839E−03	7.777E−03	**6.896E−03**
MAD	4.423E−03	**3.609E−03**	3.785E−03	3.576E−03	**2.678E−03**	3.000E−03	4.984E−03	**3.714E−03**	4.154E−03	4.762E−03	**3.541E−03**	3.998E−03	5.929E−03	5.285E−03	**4.641E−03**
Semi Dev	4.779E−03	**3.848E−03**	4.137E−03	3.954E−03	**3.020E−03**	3.371E−03	4.807E−03	**3.638E−03**	4.049E−03	4.634E−03	**3.467E−03**	3.941E−03	6.494E−03	5.762E−03	**5.137E−03**
DD	4.723E−03	**3.825E−03**	4.092E−03	3.928E−03	**3.005E−03**	3.344E−03	4.789E−03	**3.574E−03**	4.003E−03	4.608E−03	**3.413E−03**	3.899E−03	6.532E−03	5.833E−03	**5.171E−03**
Sortino ratio	**2.595E−02**	1.283E−02	2.415E−02	**1.515E−02**	1.152E−02	1.810E−02	7.393E−03	**3.571E−02**	2.251E−02	1.119E−02	**3.106E−02**	2.120E−02	0	0	0
$SR_{Std.Dev}$	**1.886E−02**	9.418E−03	1.766E−02	**1.101E−02**	8.458E−03	1.322E−02	5.225E−03	**2.523E−02**	1.589E−02	7.927E−03	**2.183E−02**	1.505E−02	0	0	0
SR_{CVaR}	**7.624E−03**	3.802E−03	7.082E−03	**4.396E−03**	3.376E−03	5.265E−03	2.334E−03	**1.121E−02**	7.102E−03	3.572E−03	**9.831E−03**	6.705E−03	0	0	0
SR_{VaR}	**1.158E−02**	5.881E−03	1.066E−02	**6.997E−03**	5.571E−03	8.789E−03	3.280E−03	**1.608E−02**	1.019E−02	5.076E−03	**1.356E−02**	9.415E−03	0	0	0
$CVaR_{97\%}$	1.905E−02	**1.526E−02**	1.653E−02	1.627E−02	**1.256E−02**	1.396E−02	1.739E−02	**1.314E−02**	1.458E−02	1.673E−02	**1.234E−02**	1.415E−02	2.568E−02	2.325E−02	**2.032E−02**
$CVaR_{95\%}$	1.608E−02	**1.291E−02**	1.396E−02	1.353E−02	**1.026E−02**	1.150E−02	1.517E−02	**1.138E−02**	1.269E−02	1.444E−02	**1.078E−02**	1.233E−02	2.162E−02	1.946E−02	**1.722E−02**
$VaR_{97\%}$	1.317E−02	**1.060E−02**	1.143E−02	1.045E−02	**7.749E−03**	9.027E−03	1.296E−02	**9.700E−03**	1.117E−02	1.221E−02	**9.194E−03**	1.067E−02	1.742E−02	1.538E−02	**1.375E−02**
$VaR_{95\%}$	1.059E−02	**8.345E−03**	9.271E−03	8.503E−03	**6.215E−03**	6.887E−03	1.079E−02	**7.934E−03**	8.839E−03	1.016E−02	**7.819E−03**	8.779E−03	1.389E−02	1.230E−02	**1.143E−02**
$RR_{97\%,97\%}$	**3.193E−04**	1.966E−04	2.348E−04	**2.351E−04**	1.343E−04	1.693E−04	**2.989E−04**	1.653E−04	2.088E−04	**2.761E−04**	1.506E−04	1.949E−04	**6.042E−04**	4.663E−04	3.617E−04
$RR_{95\%,95\%}$	**2.286E−04**	1.420E−04	1.674E−04	**1.633E−04**	9.200E−05	1.161E−04	**2.288E−04**	1.263E−04	1.589E−04	**2.078E−04**	1.154E−04	1.490E−04	**4.265E−04**	3.304E−04	2.591E−04
$VR_{97\%,97\%}$	**1.525E−04**	9.480E−05	1.109E−04	**9.830E−05**	5.560E−05	7.160E−05	**1.674E−04**	9.510E−05	1.179E−04	**1.477E−04**	8.460E−05	1.106E−04	**2.732E−04**	2.106E−04	1.659E−04
$VR_{95\%,95\%}$	**1.006E−04**	6.300E−05	7.320E−05	**6.540E−05**	3.520E−05	4.390E−05	**1.192E−04**	6.410E−05	8.080E−05	**1.058E−04**	6.000E−05	7.800E−05	**1.761E−04**	1.376E−04	1.111E−04

Table 6
Out-of-sample performance analysis of the filtered portfolios using the concatenated series with 6 months and 1 month In-sample and Out-of-sample periods respectively.

6–1	Topix			S&P Global			DAX 30			Dow Jones			Hang-seng		
	B1P	B3P	All	B1P	B3P	All	B1P	B3P	All	B1P	B3P	All	B1P	B3P	All
Mean	**7.240E−05**	4.700E−05	6.770E−05	**2.986E−04**	2.493E−04	2.580E−04	1.576E−04	**1.969E−04**	1.376E−04	**2.768E−04**	2.174E−04	2.471E−04	**1.332E−04**	6.180E−05	1.053E−04
Min	**−4.182E−02**	−4.343E−02	−4.407E−02	−5.343E−02	−4.474E−02	**−4.459E−02**	−3.854E−02	−4.011E−02	**−3.810E−02**	−5.844E−02	−5.257E−02	**−4.804E−02**	−2.808E−02	**−2.398E−02**	−2.472E−02
Max	**6.083E−02**	4.438E−02	5.199E−02	**6.415E−02**	5.580E−02	6.009E−02	4.984E−02	4.827E−02	**5.143E−02**	**7.368E−02**	5.589E−02	6.632E−02	**2.759E−02**	2.700E−02	2.693E−02
Std Dev	6.785E−03	**5.779E−03**	5.961E−03	6.583E−03	**5.302E−03**	5.592E−03	5.478E−03	**4.505E−03**	4.805E−03	7.045E−03	**5.641E−03**	5.861E−03	5.457E−03	**4.342E−03**	4.706E−03
MAD	4.648E−03	**3.861E−03**	4.018E−03	4.455E−03	**3.629E−03**	3.790E−03	3.515E−03	**2.906E−03**	2.990E−03	4.735E−03	**3.851E−03**	3.960E−03	3.948E−03	**3.087E−03**	3.349E−03
Semi Dev	4.902E−03	**4.235E−03**	4.366E−03	4.771E−03	**3.894E−03**	4.095E−03	3.951E−03	**3.223E−03**	3.481E−03	5.052E−03	**4.078E−03**	4.218E−03	3.947E−03	**3.173E−03**	3.435E−03
DD	4.868E−03	**4.213E−03**	4.335E−03	4.635E−03	**3.780E−03**	3.978E−03	3.881E−03	**3.136E−03**	3.422E−03	4.924E−03	**3.977E−03**	4.104E−03	3.881E−03	**3.143E−03**	3.384E−03
Sortino ratio	1.488E−02	1.116E−02	**1.561E−02**	6.444E−02	**6.596E−02**	6.486E−02	4.059E−02	**6.277E−02**	4.021E−02	5.620E−02	5.467E−02	**6.021E−02**	**3.431E−02**	1.968E−02	3.113E−02
$SR_{Std.Dev}$	1.068E−02	8.136E−03	**1.135E−02**	4.536E−02	**4.703E−02**	4.614E−02	2.876E−02	**4.370E−02**	2.864E−02	3.928E−02	3.854E−02	**4.216E−02**	**2.441E−02**	1.424E−02	2.239E−02
SR_{CVaR}	4.468E−03	3.320E−03	**4.631E−03**	1.897E−02	**1.930E−02**	1.888E−02	1.166E−02	**1.838E−02**	1.158E−02	1.670E−02	1.617E−02	**1.756E−02**	**1.042E−02**	5.986E−03	9.374E−03
SR_{VaR}	7.002E−03	5.424E−03	**7.447E−03**	3.028E−02	**3.059E−02**	3.035E−02	1.895E−02	**2.977E−02**	1.902E−02	2.699E−02	2.502E−02	**2.753E−02**	**1.484E−02**	8.745E−03	1.370E−02
$CVaR_{97\%}$	1.930E−02	**1.706E−02**	1.755E−02	1.887E−02	**1.541E−02**	1.632E−02	1.625E−02	**1.289E−02**	1.441E−02	2.004E−02	**1.588E−02**	1.671E−02	1.479E−02	**1.205E−02**	1.308E−02
$CVaR_{95\%}$	1.621E−02	**1.416E−02**	1.461E−02	1.574E−02	**1.292E−02**	1.366E−02	1.351E−02	**1.071E−02**	1.188E−02	1.657E−02	**1.344E−02**	1.407E−02	1.279E−02	**1.033E−02**	1.124E−02
$VaR_{97\%}$	1.306E−02	**1.102E−02**	1.163E−02	1.268E−02	**1.040E−02**	1.108E−02	1.060E−02	**8.327E−03**	9.104E−03	1.277E−02	**1.083E−02**	1.130E−02	1.096E−02	**8.637E−03**	9.379E−03
$VaR_{95\%}$	1.034E−02	**8.669E−03**	9.086E−03	9.864E−03	**8.152E−03**	8.500E−03	8.316E−03	**6.613E−03**	7.236E−03	1.025E−02	**8.689E−03**	8.975E−03	8.975E−03	**7.072E−03**	7.692E−03
$RR_{97\%,97\%}$	**3.415E−04**	2.554E−04	2.689E−04	**3.330E−04**	2.091E−04	2.363E−04	**2.489E−04**	1.624E−04	1.937E−04	**3.871E−04**	2.407E−04	2.655E−04	**2.051E−04**	1.296E−04	1.545E−04
$RR_{95\%,95\%}$	**2.446E−04**	1.799E−04	1.907E−04	**2.330E−04**	1.494E−04	1.678E−04	**1.715E−04**	1.114E−04	1.306E−04	**2.687E−04**	1.728E−04	1.892E−04	**1.533E−04**	9.620E−05	1.148E−04
$VR_{97\%,97\%}$	**1.623E−04**	1.151E−04	1.238E−04	**1.482E−04**	9.720E−05	1.087E−04	**1.025E−04**	6.680E−05	7.470E−05	**1.646E−04**	1.115E−04	1.210E−04	**1.095E−04**	6.730E−05	8.050E−05
$VR_{95\%,95\%}$	**1.046E−04**	7.420E−05	7.900E−05	**9.410E−05**	6.460E−05	7.000E−05	**6.610E−05**	4.230E−05	4.750E−05	**1.076E−04**	7.260E−05	8.080E−05	**7.530E−05**	4.790E−05	5.610E−05
	Sensex 30			CNX			IBX-50			Bovespa			Athex		
Mean	**1.431E−04**	7.370E−05	9.880E−05	**6.780E−05**	5.680E−05	6.050E−05	3.930E−05	**9.370E−05**	9.010E−05	3.080E−05	**1.002E−04**	8.260E−05	−1.313E−04	−1.127E−04	**−7.310E−05**
Min	−4.665E−02	**−3.699E−02**	−3.729E−02	−3.364E−02	**−2.980E−02**	−3.301E−02	**−3.441E−02**	−4.783E−02	−4.264E−02	**−4.065E−02**	−4.365E−02	−4.315E−02	−1.169E−01	−6.674E−02	−6.725E−02
Max	**5.900E−02**	4.903E−02	5.086E−02	4.255E−02	**4.467E−02**	4.459E−02	2.959E−02	**4.318E−02**	2.545E−02	3.153E−02	**3.468E−02**	2.455E−02	**6.624E−02**	5.054E−02	5.214E−02
Std Dev	6.299E−03	**5.250E−03**	5.596E−03	5.269E−03	**4.074E−03**	4.578E−03	6.623E−03	**5.312E−03**	5.670E−03	6.439E−03	**4.915E−03**	5.490E−03	8.901E−03	7.736E−03	**6.896E−03**
MAD	4.338E−03	**3.596E−03**	3.785E−03	3.508E−03	**2.666E−03**	3.000E−03	4.905E−03	**3.859E−03**	4.154E−03	4.736E−03	**3.578E−03**	3.998E−03	5.925E−03	5.269E−03	**4.641E−03**
Semi Dev	4.647E−03	**3.883E−03**	4.137E−03	3.843E−03	**3.013E−03**	3.371E−03	4.711E−03	**3.806E−03**	4.049E−03	4.616E−03	**3.532E−03**	3.941E−03	6.591E−03	5.695E−03	**5.137E−03**
DD	4.580E−03	**3.849E−03**	4.092E−03	3.813E−03	**2.988E−03**	3.344E−03	4.690E−03	**3.759E−03**	4.003E−03	4.600E−03	**3.482E−03**	3.899E−03	6.651E−03	5.747E−03	**5.171E−03**
Sortino ratio	**3.125E−02**	1.916E−02	2.415E−02	1.777E−02	**1.902E−02**	1.810E−02	8.372E−03	**2.492E−02**	2.251E−02	6.692E−03	**2.878E−02**	2.120E−02	0	0	0
$SR_{Std.Dev}$	**2.272E−02**	1.405E−02	1.766E−02	1.286E−02	**1.395E−02**	1.322E−02	5.929E−03	**1.763E−02**	1.589E−02	4.781E−03	**2.039E−02**	1.505E−02	0	0	0
SR_{CVaR}	**9.214E−03**	5.656E−03	7.082E−03	5.152E−03	**5.593E−03**	5.265E−03	2.655E−03	**7.850E−03**	7.102E−03	2.141E−03	**9.050E−03**	6.705E−03	0	0	0
SR_{VaR}	**1.386E−02**	8.906E−03	1.066E−02	8.278E−03	**9.390E−03**	8.789E−03	3.725E−03	**1.132E−02**	1.019E−02	3.006E−03	**1.260E−02**	9.415E−03	0	0	0
$CVaR_{97\%}$	1.833E−02	**1.549E−02**	1.653E−02	1.581E−02	**1.241E−02**	1.396E−02	1.691E−02	**1.378E−02**	1.458E−02	1.659E−02	**1.267E−02**	1.415E−02	2.603E−02	2.277E−02	**2.032E−02**
$CVaR_{95\%}$	1.553E−02	**1.304E−02**	1.396E−02	1.315E−02	**1.016E−02**	1.150E−02	1.479E−02	**1.193E−02**	1.269E−02	1.438E−02	**1.107E−02**	1.233E−02	2.180E−02	1.923E−02	**1.722E−02**
$VaR_{97\%}$	1.268E−02	**1.067E−02**	1.143E−02	1.039E−02	**7.739E−03**	9.027E−03	1.286E−02	**1.026E−02**	1.117E−02	1.212E−02	**9.439E−03**	1.067E−02	1.746E−02	1.598E−02	**1.375E−02**
$VaR_{95\%}$	1.032E−02	**8.280E−03**	9.271E−03	8.184E−03	**6.052E−03**	6.887E−03	1.054E−02	**8.277E−03**	8.839E−03	1.024E−02	**7.954E−03**	8.779E−03	1.389E−02	1.258E−02	**1.143E−02**
$RR_{97\%,97\%}$	**2.925E−04**	2.038E−04	2.348E−04	**2.214E−04**	1.321E−04	1.693E−04	**2.850E−04**	1.800E−04	2.088E−04	**2.674E−04**	1.519E−04	1.949E−04	**6.097E−04**	4.597E−04	3.617E−04
$RR_{95\%,95\%}$	**2.120E−04**	1.469E−04	1.674E−04	**1.546E−04**	9.050E−05	1.161E−04	**2.170E−04**	1.368E−04	1.589E−04	**2.021E−04**	1.169E−04	1.490E−04	**4.272E−04**	3.279E−04	2.591E−04
$VR_{97\%,97\%}$	**1.414E−04**	9.850E−05	1.109E−04	**9.590E−05**	5.550E−05	7.160E−05	**1.624E−04**	1.018E−04	1.179E−04	**1.458E−04**	8.700E−05	1.106E−04	**2.672E−04**	2.146E−04	1.659E−04
$VR_{95\%,95\%}$	**9.690E−05**	6.440E−05	7.320E−05	**6.240E−05**	3.370E−05	4.390E−05	**1.118E−04**	6.950E−05	8.080E−05	**1.038E−04**	6.240E−05	7.800E−05	**1.727E−04**	1.408E−04	1.111E−04

Table 7
Out-of-sample performance analysis of the filtered portfolios using the concatenated series with 3 months and 1 month In-sample and Out-of-sample periods respectively.

3–1	Topix			S&P Global			DAX 30			Dow Jones			Hang-seng		
	B1P	B3P	All	B1P	B3P	All	B1P	B3P	All	B1P	B3P	All	B1P	B3P	All
Mean	3.680E−05	**8.270E−05**	6.680E−05	2.827E−04	**2.930E−04**	2.611E−04	**1.682E−04**	1.443E−04	1.355E−04	2.597E−04	**2.635E−04**	2.517E−04	**9.610E−05**	8.240E−05	9.490E−05
Min	−4.262E−02	−4.526E−02	−4.407E−02	−5.698E−02	−4.091E−02	−4.459E−02	−4.723E−02	−3.212E−02	−3.810E−02	−6.280E−02	−4.672E−02	−4.804E−02	−2.948E−02	−2.353E−02	−2.472E−02
Max	5.663E−02	4.782E−02	5.199E−02	6.407E−02	4.831E−02	6.009E−02	4.908E−02	4.568E−02	5.143E−02	7.787E−02	4.652E−02	6.632E−02	3.114E−02	2.605E−02	2.693E−02
Std Dev	6.681E−03	**5.752E−03**	5.925E−03	6.477E−03	**5.261E−03**	5.563E−03	5.524E−03	**4.412E−03**	4.782E−03	6.894E−03	**5.615E−03**	5.831E−03	5.400E−03	**4.368E−03**	4.733E−03
MAD	4.554E−03	**3.852E−03**	3.986E−03	4.373E−03	**3.653E−03**	3.774E−03	3.510E−03	**2.875E−03**	2.983E−03	4.660E−03	**3.834E−03**	3.944E−03	3.895E−03	**3.097E−03**	3.376E−03
Semi Dev	4.836E−03	**4.225E−03**	4.340E−03	4.698E−03	**3.860E−03**	4.075E−03	3.980E−03	**3.167E−03**	3.464E−03	4.917E−03	**4.075E−03**	4.200E−03	3.892E−03	**3.178E−03**	3.454E−03
DD	4.819E−03	**4.187E−03**	4.309E−03	4.569E−03	**3.724E−03**	3.957E−03	3.907E−03	**3.102E−03**	3.407E−03	4.796E−03	**3.954E−03**	4.084E−03	3.845E−03	**3.138E−03**	3.408E−03
Sortino ratio	7.628E−03	**1.975E−02**	1.550E−02	6.189E−02	**7.867E−02**	6.598E−02	4.306E−02	**4.650E−02**	3.977E−02	5.415E−02	**6.665E−02**	6.163E−02	2.500E−02	2.627E−02	**2.784E−02**
$SR_{Std.Dev}$	5.502E−03	**1.438E−02**	1.128E−02	4.365E−02	**5.569E−02**	4.693E−02	3.046E−02	**3.270E−02**	2.833E−02	3.767E−02	**4.693E−02**	4.316E−02	1.780E−02	1.887E−02	**2.005E−02**
SR_{CVaR}	2.283E−03	**5.865E−03**	4.592E−03	1.810E−02	**2.335E−02**	1.920E−02	1.244E−02	**1.350E−02**	1.147E−02	1.607E−02	**1.968E−02**	1.796E−02	7.661E−03	7.936E−03	**8.403E−03**
SR_{VaR}	3.501E−03	**9.493E−03**	7.383E−03	2.911E−02	**3.642E−02**	3.085E−02	2.083E−02	**2.151E−02**	1.896E−02	2.619E−02	**3.100E−02**	2.820E−02	1.058E−02	1.151E−02	**1.220E−02**
$CVaR_{97\%}$	1.915E−02	**1.702E−02**	1.747E−02	1.861E−02	**1.502E−02**	1.623E−02	1.642E−02	**1.280E−02**	1.431E−02	1.935E−02	**1.598E−02**	1.662E−02	1.442E−02	**1.217E−02**	1.309E−02
$CVaR_{95\%}$	1.610E−02	**1.410E−02**	1.455E−02	1.562E−02	**1.255E−02**	1.360E−02	1.353E−02	**1.069E−02**	1.181E−02	1.616E−02	**1.339E−02**	1.401E−02	1.255E−02	**1.039E−02**	1.129E−02
$VaR_{97\%}$	1.281E−02	**1.103E−02**	1.157E−02	1.254E−02	**9.773E−03**	1.106E−02	1.053E−02	**8.360E−03**	9.069E−03	1.279E−02	**1.044E−02**	1.123E−02	1.075E−02	**8.690E−03**	9.385E−03
$VaR_{95\%}$	1.050E−02	**8.712E−03**	9.048E−03	9.713E−03	**8.043E−03**	8.462E−03	8.077E−03	**6.707E−03**	7.147E−03	9.918E−03	**8.502E−03**	8.924E−03	9.083E−03	**7.161E−03**	7.779E−03
$RR_{97\%,97\%}$	**3.319E−04**	2.550E−04	2.661E−04	**3.218E−04**	2.050E−04	2.336E−04	**2.553E−04**	1.573E−04	1.910E−04	**3.637E−04**	2.452E−04	2.626E−04	**1.975E−04**	1.354E−04	1.545E−04
$RR_{95\%,95\%}$	**2.383E−04**	1.786E−04	1.889E−04	**2.266E−04**	1.457E−04	1.661E−04	**1.738E−04**	1.087E−04	1.291E−04	**2.560E−04**	1.745E−04	1.873E−04	**1.482E−04**	9.980E−05	1.153E−04
$VR_{97\%,97\%}$	**1.542E−04**	1.115E−04	1.227E−04	**1.444E−04**	9.240E−05	1.082E−04	**1.030E−04**	6.720E−05	7.420E−05	**1.619E−04**	1.104E−04	1.197E−04	**1.058E−04**	7.080E−05	8.060E−05
$VR_{95\%,95\%}$	**1.069E−04**	7.240E−05	7.860E−05	**9.150E−05**	6.380E−05	6.930E−05	**6.300E−05**	4.120E−05	4.660E−05	**1.041E−04**	7.420E−05	7.850E−05	**7.450E−05**	4.910E−05	5.670E−05
	Sensex 30			CNX			IBX-50			Bovespa			Athex		
Mean	**1.696E−04**	6.590E−05	1.028E−04	6.500E−05	**7.210E−05**	6.170E−05	6.980E−05	4.500E−05	**8.070E−05**	5.480E−05	4.900E−05	**7.250E−05**	−1.531E−04	−1.115E−04	**−6.963E−05**
Min	−4.340E−02	−3.547E−02	−3.729E−02	−3.613E−02	−2.945E−02	−3.301E−02	−4.377E−02	−4.085E−02	−4.264E−02	−4.544E−02	−4.303E−02	−4.315E−02	−1.051E−01	−6.503E−02	−6.725E−02
Max	5.451E−02	5.763E−02	5.086E−02	4.830E−02	4.164E−02	4.459E−02	4.702E−02	2.112E−02	2.545E−02	3.450E−02	2.098E−02	2.455E−02	**7.340E−02**	4.729E−02	5.214E−02
Std Dev	6.173E−03	**5.311E−03**	5.561E−03	5.264E−03	**3.999E−03**	4.549E−03	6.642E−03	**5.181E−03**	5.668E−03	6.345E−03	**4.956E−03**	5.484E−03	8.650E−03	7.824E−03	**6.847E−03**
MAD	4.249E−03	**3.628E−03**	3.760E−03	3.421E−03	**2.664E−03**	2.981E−03	4.870E−03	**3.816E−03**	4.153E−03	4.658E−03	**3.610E−03**	3.994E−03	5.785E−03	5.267E−03	**4.601E−03**
Semi Dev	4.515E−03	**3.940E−03**	4.113E−03	3.817E−03	**2.955E−03**	3.350E−03	4.695E−03	**3.745E−03**	4.046E−03	4.534E−03	**3.599E−03**	3.936E−03	6.405E−03	5.804E−03	**5.102E−03**
DD	4.436E−03	**3.910E−03**	4.066E−03	3.788E−03	**2.922E−03**	3.322E−03	4.659E−03	**3.722E−03**	4.005E−03	4.506E−03	**3.574E−03**	3.899E−03	6.474E−03	5.854E−03	**5.134E−03**
Sortino ratio	**3.825E−02**	1.685E−02	2.528E−02	1.715E−02	**2.468E−02**	1.858E−02	1.498E−02	1.208E−02	**2.015E−02**	1.215E−02	1.372E−02	**1.859E−02**	0	0	0
$SR_{Std.Dev}$	**2.748E−02**	1.240E−02	1.848E−02	1.234E−02	**1.803E−02**	1.357E−02	1.051E−02	8.676E−03	**1.424E−02**	8.631E−03	9.896E−03	**1.321E−02**	0	0	0
SR_{CVaR}	**1.126E−02**	4.995E−03	7.408E−03	4.994E−03	**7.254E−03**	5.406E−03	4.802E−03	3.804E−03	**6.357E−03**	3.915E−03	4.345E−03	**5.886E−03**	0	0	0
SR_{VaR}	**1.654E−02**	7.720E−03	1.117E−02	8.437E−03	**1.179E−02**	9.041E−03	6.661E−03	5.213E−03	**9.070E−03**	5.468E−03	6.166E−03	**8.251E−03**	0	0	0
$CVaR_{97\%}$	1.763E−02	**1.562E−02**	1.644E−02	1.584E−02	**1.200E−02**	1.386E−02	1.669E−02	**1.353E−02**	1.452E−02	1.608E−02	**1.306E−02**	1.407E−02	2.551E−02	2.332E−02	**2.020E−02**
$CVaR_{95\%}$	1.507E−02	**1.319E−02**	1.388E−02	1.301E−02	**9.943E−03**	1.142E−02	1.453E−02	**1.182E−02**	1.269E−02	1.399E−02	**1.129E−02**	1.231E−02	2.154E−02	1.975E−02	**1.713E−02**
$VaR_{97\%}$	1.258E−02	**1.053E−02**	1.128E−02	9.728E−03	**7.548E−03**	8.974E−03	1.232E−02	**9.936E−03**	1.129E−02	1.178E−02	**9.277E−03**	1.072E−02	1.729E−02	1.606E−02	**1.355E−02**
$VaR_{95\%}$	1.026E−02	**8.531E−03**	9.204E−03	7.699E−03	**6.117E−03**	6.829E−03	1.048E−02	**8.623E−03**	8.897E−03	1.002E−02	**7.955E−03**	8.782E−03	1.369E−02	1.261E−02	**1.135E−02**
$RR_{97\%,97\%}$	**2.873E−04**	2.038E−04	2.320E−04	**2.256E−04**	1.251E−04	1.671E−04	**2.851E−04**	1.707E−04	2.080E−04	**2.582E−04**	1.572E−04	1.933E−04	**5.731E−04**	4.707E−04	3.575E−04
$RR_{95\%,95\%}$	**2.076E−04**	1.471E−04	1.655E−04	**1.533E−04**	8.720E−05	1.146E−04	**2.162E−04**	1.320E−04	1.592E−04	**1.958E−04**	1.208E−04	1.487E−04	**4.075E−04**	3.386E−04	2.562E−04
$VR_{97\%,97\%}$	**1.417E−04**	9.480E−05	1.089E−04	**8.750E−05**	5.340E−05	7.090E−05	**1.592E−04**	9.590E−05	1.196E−04	**1.391E−04**	8.740E−05	1.114E−04	**2.643E−04**	2.184E−04	1.627E−04
$VR_{95\%,95\%}$	**9.170E−05**	6.580E−05	7.220E−05	**5.630E−05**	3.460E−05	4.340E−05	**1.125E−04**	7.220E−05	8.130E−05	**1.022E−04**	6.260E−05	7.800E−05	**1.667E−04**	1.439E−04	1.095E−04

Table 8
Out-of-sample performance analysis of the proposed EI model using the concatenated series.

	ETDA1	ETDA2	EIM1	EIM2	EIM3	ETDA1	ETDA2	EIM1	EIM2	EIM3	ETDA1	ETDA2	EIM1	EIM2	EIM3	ETDA1	ETDA2	EIM1	EIM2	EIM3
Topix	6–3					6–2						6-1				3-1				
Mean	**1.25E−04**	6.27E−05	5.20E−05	*1.13E−05*	1.61E−05	**1.40E−04**	−1.21E−04	3.83E−06	5.04E−06	−1.07E−05	**3.02E−04**	7.46E−05	−6.21E−05	−1.62E−06	−3.85E−06	**3.20E−04**	1.26E−04	−1.07E−04	−1.35E−05	−1.42E−05
Min	−6.95E−02	−7.13E−02	−5.18E−02	−4.74E−02	−4.93E−02	−8.07E−02	−7.13E−02	−6.95E−02	−4.65E−02	−4.64E−02	−8.05E−02	−7.13E−02	−6.95E−02	−4.68E−02	−4.69E−02	−6.76E−02	−7.14E−02	−5.87E−02	−4.64E−02	−4.71E−02
Max	7.26E−02	**7.53E−02**	6.36E−02	5.64E−02	5.75E−02	**7.63E−02**	6.06E−02	6.39E−02	5.65E−02	5.67E−02	**7.63E−02**	6.06E−02	6.39E−02	5.65E−02	5.67E−02	**7.64E−02**	4.92E−02	4.84E−02	5.53E−02	5.43E−02
Std Dev	9.73E−03	9.84E−03	7.56E−03	**6.36E−03**	6.42E−03	9.74E−03	9.74E−03	7.43E−03	**6.35E−03**	6.38E−03	9.67E−03	9.54E−03	7.35E−03	**6.35E−03**	6.38E−03	9.65E−03	9.16E−03	6.94E−03	**6.31E−03**	6.32E−03
Skewness	−1.61E−03	1.61E−01	−1.73E−02	−2.74E−01	−2.81E−01	2.97E−02	−2.16E−01	−2.80E−01	−2.74E−01	−2.63E−01	1.80E−01	4.14E−02	−4.19E−01	−2.77E−01	−2.54E−01	2.23E−01	−2.72E−01	−4.57E−01	−3.27E−01	−3.22E−01
Kurtosis	7.04E+00	7.34E+00	7.27E+00	7.33E+00	7.61E+00	6.68E+00	5.50E+00	1.01E+01	7.22E+00	7.16E+00	6.69E+00	5.79E+00	1.09E+01	7.13E+00	7.04E+00	6.18E+00	4.09E+00	7.50E+00	7.15E+00	7.10E+00
EMR	**1.12E−04**	4.90E−05	3.84E−05	−2.36E−06	2.50E−06	**1.34E−04**	−1.27E−04	−2.11E−06	−8.99E−07	−1.67E−05	**2.96E−04**	6.87E−05	−6.80E−05	−7.57E−06	−9.79E−06	**3.16E−04**	1.22E−04	−1.10E−04	−1.68E−05	−1.76E−05
$SR_{Std.Dev}$	**1.29E−02**	6.37E−03	6.88E−03	1.77E−03	2.51E−03	**1.44E−02**	−1.24E−02	5.16E−04	7.94E−04	−1.68E−03	**3.12E−02**	7.82E−03	−8.44E−03	−2.56E−04	−6.03E−04	**3.31E−02**	1.37E−02	−1.54E−02	−2.14E−03	−2.25E−03
Sortino ratio	**2.34E−02**	9.02E−03	1.36E−02	−5.63E−03	5.30E−03	**2.78E−02**	−2.22E−02	−7.14E−04	−2.24E−03	−3.61E−02	**6.43E−02**	1.29E−02	−2.35E−02	−1.87E−02	−2.11E−02	**6.59E−02**	2.41E−02	−3.69E−02	−3.11E−02	−3.21E−02
$RR_{95\%,95\%}$	2.65E−04	**3.42E−04**	9.34E−05	2.01E−06	2.55E−06	2.76E−04	**3.43E−04**	9.54E−05	1.83E−06	2.28E−06	2.67E−04	**3.35E−04**	8.39E−05	1.77E−06	2.33E−06	**2.84E−04**	**2.84E−04**	9.12E−05	2.96E−06	2.86E−06
$RR_{97\%,97\%}$	3.68E−04	**4.89E−04**	1.34E−04	2.91E−06	3.73E−06	3.83E−04	**4.79E−04**	1.36E−04	2.60E−06	3.21E−06	3.71E−04	**4.70E−04**	1.21E−04	2.52E−06	3.32E−06	**4.03E−04**	3.93E−04	1.31E−04	4.13E−06	4.03E−06
$VaRR_{95\%,95\%}$	1.19E−04	**1.40E−04**	3.67E−05	7.51E−07	1.00E−06	1.19E−04	**1.50E−04**	3.72E−05	7.52E−07	9.74E−07	1.16E−04	**1.40E−04**	3.28E−05	7.26E−07	9.63E−07	1.17E−04	**1.24E−04**	3.66E−05	1.26E−06	1.23E−06
$VaRR_{97\%,97\%}$	1.77E−04	**2.12E−04**	5.99E−05	1.27E−06	1.54E−06	1.87E−04	**2.22E−04**	6.23E−05	1.19E−06	1.47E−06	1.77E−04	**2.17E−04**	5.39E−05	1.12E−06	1.44E−06	1.78E−04	**1.90E−04**	5.70E−05	1.98E−06	1.85E−06
DAX 30	6−3					6−2						6−1				3−1				
Mean	2.08E−04	**2.53E−04**	1.37E−04	7.16E−05	6.34E−05	2.09E−04	**3.66E−04**	1.39E−04	6.23E−05	5.74E−05	−5.73E−05	**3.65E−04**	7.02E−05	6.04E−05	1.74E−05	5.65E−06	**1.95E−04**	7.56E−05	5.40E−05	4.53E−05
Min	−2.04E−01	−1.21E−01	−5.47E−02	−3.93E−02	−4.16E−02	−2.04E−01	−4.17E−02	−5.24E−02	−3.93E−02	−4.16E−02	−2.04E−01	−5.21E−02	−5.24E−02	−3.62E−02	−4.19E−02	−2.05E−01	−7.19E−02	−6.47E−02	−3.77E−02	−4.85E−02
Max	**8.68E−02**	7.42E−02	6.88E−02	4.88E−02	4.84E−02	6.98E−02	6.74E−02	**8.94E−02**	4.93E−02	5.14E−02	7.01E−02	5.84E−02	**8.94E−02**	4.93E−02	5.14E−02	7.01E−02	**7.02E−02**	5.92E−02	4.14E−02	4.48E−02
Std Dev	1.20E−02	1.03E−02	6.97E−03	**5.67E−03**	5.86E−03	1.06E−02	9.01E−03	7.21E−03	**5.61E−03**	5.84E−03	1.16E−02	9.21E−03	7.11E−03	**5.60E−03**	5.87E−03	1.15E−02	1.01E−02	6.48E−03	**5.58E−03**	5.77E−03
Skewness	−1.41E+00	−9.16E−01	1.85E−01	−3.62E−01	−3.94E−01	−1.59E+00	4.03E−01	6.06E−01	−3.51E−01	−2.87E−01	−2.69E+00	1.10E−01	6.09E−01	−2.72E−01	−4.56E−01	−2.82E+00	1.11E−01	−5.46E−01	−3.24E−01	−5.09E−01
Kurtosis	3.51E+01	1.47E+01	1.36E+01	6.53E+00	7.58E+00	4.48E+01	4.95E+00	1.78E+01	6.73E+00	8.32E+00	4.49E+01	4.04E+00	1.80E+01	6.59E+00	8.42E+00	4.61E+01	7.18E+00	1.25E+01	5.13E+00	7.87E+00
EMR	8.66E−05	**1.31E−04**	1.61E−05	−4.95E−05	−5.77E−05	9.33E−05	**2.50E−04**	2.30E−05	−5.39E−05	−5.87E−05	−1.73E−04	**2.49E−04**	−4.59E−05	−5.58E−05	−9.87E−05	−1.14E−04	**7.56E−05**	−4.43E−05	−6.58E−05	−7.45E−05
$SR_{Std.Dev}$	1.74E−02	**2.46E−02**	1.97E−02	1.26E−02	1.08E−02	1.97E−02	**4.06E−02**	1.93E−02	1.11E−02	9.83E−03	−4.94E−03	**3.97E−02**	9.89E−03	1.08E−02	2.97E−03	4.93E−04	**1.93E−02**	1.17E−02	9.68E−03	7.87E−03
Sortino ratio	1.16E−02	**2.12E−02**	4.89E−03	−2.75E−02	−2.69E−02	1.42E−02	**4.97E−02**	6.81E−03	−2.95E−02	−2.69E−02	−2.17E−02	**4.92E−02**	−1.39E−02	−3.20E−02	−4.44E−02	−1.45E−02	**1.34E−02**	−1.30E−02	−3.55E−02	−3.42E−02
$RR_{95\%,95\%}$	**5.51E−04**	3.93E−04	1.19E−04	2.74E−05	4.34E−05	**4.10E−04**	3.12E−04	1.29E−04	2.81E−05	4.56E−05	**5.00E−04**	3.13E−04	1.21E−04	2.63E−05	4.40E−05	**4.69E−04**	3.84E−04	1.06E−04	3.04E−05	4.11E−05
$RR_{97\%,97\%}$	**8.95E−04**	5.79E−04	1.85E−04	4.07E−05	6.96E−05	**6.34E−04**	4.50E−04	2.01E−04	4.18E−05	7.22E−05	**8.03E−04**	4.44E−04	1.90E−04	3.89E−05	6.91E−05	**7.42E−04**	5.68E−04	1.56E−04	4.44E−05	6.32E−05
$VaRR_{95\%,95\%}$	**1.45E−04**	1.44E−04	3.91E−05	9.91E−06	1.22E−05	**1.30E−04**	1.25E−04	3.95E−05	1.01E−05	1.36E−05	**1.37E−04**	1.30E−04	3.67E−05	9.84E−06	1.35E−05	1.38E−04	**1.41E−04**	3.84E−05	1.18E−05	1.31E−05
$VaRR_{97\%,97\%}$	**2.54E−04**	2.32E−04	6.09E−05	1.60E−05	2.15E−05	**2.20E−04**	1.97E−04	6.50E−05	1.64E−05	2.24E−05	**2.45E−04**	2.03E−04	6.20E−05	1.56E−05	2.19E−05	**2.33E−04**	2.18E−04	6.26E−05	1.81E−05	2.16E−05
Dow Jones	6−3					6−2						6−1				3−1				
Mean	**2.43E−04**	6.97E−05	2.22E−04	1.17E−04	1.18E−04	**1.61E−04**	1.12E−04	1.36E−04	1.21E−04	1.19E−04	6.72E−05	−1.21E−04	**2.02E−04**	1.17E−04	1.20E−04	1.02E−04	**1.64E−04**	1.21E−04	1.22E−04	1.27E−04
Min	−8.56E−02	−8.57E−02	−3.50E−02	−3.70E−02	−3.78E−02	−8.40E−02	−5.79E−02	−6.51E−02	−3.68E−02	−3.82E−02	−8.40E−02	−6.98E−02	−6.51E−02	−3.68E−02	−3.82E−02	−8.40E−02	−7.12E−02	−4.08E−02	−3.65E−02	−3.77E−02
Max	**8.12E−02**	5.66E−02	4.88E−02	4.77E−02	4.79E−02	9.52E−02	8.23E−02	**9.69E−02**	4.70E−02	4.75E−02	9.52E−02	8.23E−02	**9.69E−02**	4.70E−02	4.75E−02	**9.51E−02**	6.17E−02	7.15E−02	4.18E−02	4.26E−02
Std Dev	8.82E−03	7.51E−03	4.90E−03	**4.73E−03**	4.74E−03	8.64E−03	7.34E−03	5.87E−03	**4.73E−03**	4.77E−03	8.94E−03	7.47E−03	5.69E−03	**4.72E−03**	4.73E−03	8.95E−03	8.01E−03	5.07E−03	**4.66E−03**	4.68E−03
Skewness	1.64E−01	−6.77E−01	3.53E−01	−9.31E−02	−6.70E−02	7.16E−02	2.67E−01	4.00E−01	−9.82E−02	−5.49E−02	9.01E−02	−4.89E−01	1.12E+00	−1.20E−01	−1.30E−01	1.24E−01	−3.22E−01	5.67E−01	−2.15E−01	−2.15E−01
Kurtosis	1.60E+01	1.47E+01	1.29E+01	1.21E+01	1.28E+01	1.82E+01	1.39E+01	4.82E+01	1.20E+01	1.31E+01	1.71E+01	1.50E+01	5.19E+01	1.19E+01	1.25E+01	1.65E+01	1.12E+01	2.21E+01	1.04E+01	1.11E+01
EMR	**1.26E−04**	−4.71E−05	1.06E−04	3.69E−07	1.52E−06	**4.60E−05**	−3.08E−06	2.10E−05	5.59E−06	4.09E−06	−4.77E−05	−2.36E−04	**8.74E−05**	2.04E−06	5.36E−06	−9.45E−06	**5.22E−05**	9.40E−06	9.90E−06	1.52E−05
SH_{StdDev}	2.75E−02	9.28E−03	**4.54E−02**	2.48E−02	2.50E−02	1.86E−02	1.52E−02	**2.31E−02**	2.55E−02	2.50E−02	7.52E−03	−1.62E−02	**3.56E−02**	2.48E−02	2.54E−02	1.14E−02	2.05E−02	2.39E−02	2.61E−02	**2.71E−02**
Sortino ratio	2.77E−02	−1.12E−02	**5.52E−02**	9.71E−04	3.36E−03	1.04E−02	−8.16E−04	7.11E−03	**1.49E−02**	9.04E−03	−1.01E−02	−5.59E−02	**3.54E−02**	5.39E−03	1.20E−02	−2.06E−03	1.24E−02	4.33E−03	2.01E−02	**3.01E−02**
$RR_{95\%,95\%}$	**2.59E−04**	1.91E−04	4.50E−05	1.69E−06	2.36E−06	**2.24E−04**	1.56E−04	7.72E−05	1.67E−06	2.46E−06	**2.45E−04**	1.69E−04	6.20E−05	1.62E−06	2.37E−06	**2.39E−04**	1.92E−04	5.57E−05	2.75E−06	3.00E−06
$RR_{97\%,97\%}$	**4.11E−04**	2.94E−04	6.21E−05	2.43E−06	3.40E−06	**3.52E−04**	2.30E−04	1.24E−04	2.42E−06	3.51E−06	**3.82E−04**	2.48E−04	9.58E−05	2.32E−06	3.44E−06	**3.68E−04**	2.86E−04	8.12E−05	3.90E−06	4.29E−06
$VaRR_{95\%,95\%}$	**7.64E−05**	6.21E−05	2.01E−05	6.47E−07	9.34E−07	**6.93E−05**	6.15E−05	2.27E−05	6.39E−07	9.94E−07	**7.73E−05**	6.31E−05	2.04E−05	6.47E−07	8.84E−07	**7.72E−05**	6.95E−05	2.06E−05	1.16E−06	1.21E−06
$VaRR_{97\%,97\%}$	**1.27E−04**	1.01E−04	3.03E−05	1.10E−06	1.50E−06	**1.11E−04**	9.07E−05	3.40E−05	1.01E−06	1.57E−06	**1.22E−04**	9.78E−05	3.11E−05	1.04E−06	1.52E−06	**1.26E−04**	1.09E−04	3.28E−05	1.69E−06	1.89E−06

Table 9
Out-of-sample performance analysis of the proposed EI model using the concatenated series.

	ETDA1	ETDA2	EIM1	EIM2	EIM3	ETDA1	ETDA2	EIM1	EIM2	EIM3	ETDA1	ETDA2	EIM1	EIM2	EIM3	ETDA1	ETDA2	EIM1	EIM2	EIM3
S&P Global	6–3					6–2					6-1					3-1				
Mean	**2.66E−04**	4.52E−05	1.73E−04	6.65E−05	5.93E−05	2.03E−04	1.31E−04	**2.22E−04**	6.95E−05	6.30E−05	**2.54E−04**	1.14E−04	2.07E−04	5.96E−05	6.76E−05	2.03E−04	**2.86E−04**	1.24E−04	7.93E−05	6.54E−05
Min	−2.11E−01	−8.54E−02	−6.70E−02	−3.21E−02	−3.17E−02	−9.74E−02	−5.79E−02	−6.23E−02	−3.02E−02	−3.17E−02	−2.11E−01	−5.79E−02	−6.23E−02	−3.02E−02	−3.17E−02	−2.11E−01	−5.32E−02	−6.48E−02	−2.90E−02	−2.86E−02
Max	**1.38E−01**	7.53E−02	7.67E−02	4.14E−02	4.49E−02	**1.13E−01**	4.95E−02	8.50E−02	4.46E−02	4.39E−02	**1.38E−01**	9.50E−02	8.50E−02	4.46E−02	4.39E−02	**1.38E−01**	9.50E−02	5.83E−02	4.10E−02	4.45E−02
Std Dev	1.31E−02	9.25E−03	6.66E−03	**4.43E−03**	4.47E−03	1.15E−02	8.76E−03	6.67E−03	**4.45E−03**	4.48E−03	1.30E−02	9.24E−03	6.63E−03	**4.42E−03**	4.44E−03	1.29E−02	9.63E−03	5.85E−03	**4.42E−03**	4.49E−03
Skewness	−5.02E−01	−4.69E−02	6.04E−01	−3.33E−01	−3.11E−01	5.30E−01	−1.52E−01	2.01E−01	−2.79E−01	−2.83E−01	−9.71E−01	2.02E−01	3.70E−01	−2.38E−01	−2.82E−01	−1.14E+00	4.47E−01	−3.24E−01	−2.90E−01	−2.04E−01
Kurtosis	3.48E+01	6.60E+00	2.82E+01	8.24E+00	1.01E+01	1.41E+01	3.85E+00	2.74E+01	8.92E+00	9.69E+00	3.47E+01	6.49E+00	2.80E+01	9.02E+00	9.81E+00	3.36E+01	8.40E+00	1.68E+01	8.29E+00	9.02E+00
EMR	**2.04E−04**	−1.63E−05	1.11E−04	4.97E−06	−2.20E−06	1.45E−04	7.24E−05	**1.63E−04**	1.10E−05	4.52E−06	**1.95E−04**	5.52E−05	1.48E−04	1.10E−06	9.16E−06	1.47E−04	**2.30E−04**	6.86E−05	2.34E−05	9.49E−06
$SR_{Std.Dev}$	2.03E−02	4.89E−03	**2.59E−02**	1.50E−02	1.33E−02	1.77E−02	1.49E−02	**3.33E−02**	1.56E−02	1.41E−02	1.96E−02	1.23E−02	**3.12E−02**	1.35E−02	1.52E−02	1.58E−02	**2.97E−02**	2.13E−02	1.80E−02	1.45E−02
Sortino ratio	2.60E−02	−2.84E−03	**3.59E−02**	5.02E−03	−2.20E−03	2.22E−02	1.30E−02	**5.20E−02**	1.14E−02	4.70E−03	2.44E−02	9.73E−03	**4.71E−02**	1.14E−03	9.49E−03	1.83E−02	**4.07E−02**	2.31E−02	2.13E−02	8.36E−03
$RR_{95\%,95\%}$	**6.45E−04**	3.62E−04	1.18E−04	9.32E−06	9.70E−06	**5.19E−04**	3.33E−04	1.22E−04	9.05E−06	9.20E−06	**6.50E−04**	3.61E−04	1.19E−04	9.12E−06	9.50E−06	**6.45E−04**	3.80E−04	9.93E−05	1.20E−05	1.31E−05
$RR_{97\%,97\%}$	**1.03E−03**	5.02E−04	1.90E−04	1.25E−05	1.30E−05	**7.88E−04**	4.48E−04	1.95E−04	1.21E−05	1.23E−05	**1.02E−03**	4.95E−04	1.89E−04	1.21E−05	1.27E−05	**9.96E−04**	5.35E−04	1.47E−04	1.58E−05	1.75E−05
$VaRR_{95\%,95\%}$	**1.93E−04**	1.59E−04	3.26E−05	4.47E−06	4.67E−06	**1.78E−04**	1.57E−04	3.53E−05	4.40E−06	4.55E−06	**2.00E−04**	1.66E−04	3.52E−05	4.57E−06	4.64E−06	**2.07E−04**	1.59E−04	3.56E−05	6.07E−06	6.37E−06
$VaRR_{97\%,97\%}$	**3.10E−04**	2.41E−04	5.74E−05	6.67E−06	6.91E−06	**2.81E−04**	2.30E−04	5.76E−05	6.55E−06	6.65E−06	**3.26E−04**	2.50E−04	5.71E−05	6.60E−06	6.88E−06	**3.29E−04**	2.42E−04	5.72E−05	8.68E−06	9.25E−06
Hang−seng	6−3					6−2					6−1					3−1				
Mean	**2.76E−04**	2.75E−04	9.95E−05	4.41E−05	4.08E−05	**4.05E−04**	3.56E−04	−5.22E−05	2.85E−05	3.22E−05	2.03E−04	**2.80E−04**	−2.75E−05	2.43E−05	2.17E−05	2.30E−04	**5.20E−04**	6.76E−05	3.03E−05	6.19E−05
Min	−5.18E−02	−9.10E−02	−3.59E−02	−2.64E−02	−2.63E−02	−9.70E−02	−1.20E−01	−4.07E−02	−2.64E−02	−2.63E−02	−9.70E−02	−1.20E−01	−4.07E−02	−2.64E−02	−2.66E−02	−9.70E−02	−5.86E−02	−3.32E−02	−2.43E−02	−2.67E−02
Max	4.83E−02	**7.30E−02**	2.03E−02	2.44E−02	2.51E−02	6.18E−02	**6.95E−02**	5.56E−02	2.52E−02	2.59E−02	6.18E−02	**7.30E−02**	5.56E−02	2.44E−02	2.51E−02	6.18E−02	**7.63E−02**	2.24E−02	2.44E−02	2.43E−02
Std Dev	9.88E−03	1.18E−02	4.99E−03	4.78E−03	**4.77E−03**	1.05E−02	1.14E−02	5.35E−03	**4.83E−03**	4.85E−03	1.05E−02	1.15E−02	5.39E−03	4.83E−03	**4.82E−03**	1.06E−02	1.07E−02	5.05E−03	4.84E−03	**4.82E−03**
Skewness	2.85E−02	−1.34E−01	−6.21E−01	−3.36E−01	−3.30E−01	−1.20E−01	−3.50E−01	−3.26E−01	−3.48E−01	−3.41E−01	1.64E−01	−3.34E−01	−2.99E−01	−3.36E−01	−3.43E−01	1.26E−01	4.30E−01	−2.82E−01	−3.65E−01	−3.17E−01
Kurtosis	2.40E+00	5.95E+00	5.12E+00	3.02E+00	3.09E+00	5.21E+00	8.77E+00	1.09E+01	2.93E+00	3.05E+00	5.65E+00	1.02E+01	1.09E+01	2.83E+00	2.97E+00	5.25E+00	3.58E+00	4.46E+00	2.53E+00	2.41E+00
EMR	**2.28E−04**	2.27E−04	5.09E−05	−4.52E−06	−7.82E−06	**3.63E−04**	3.14E−04	−9.42E−05	−1.35E−05	−9.80E−06	1.61E−04	**2.38E−04**	−6.95E−05	−1.78E−05	−2.03E−05	1.94E−04	**4.84E−04**	3.15E−05	−5.81E−06	2.58E−05
$SR_{Std.Dev}$	**2.80E−02**	2.33E−02	2.00E−02	9.24E−03	8.56E−03	**3.85E−02**	3.13E−02	−9.75E−03	5.90E−03	6.64E−03	1.94E−02	**2.42E−02**	−5.11E−03	5.02E−03	4.51E−03	**2.17E−02**	4.86E−02	1.34E−02	6.25E−03	1.28E−02
Sortino ratio	**4.16E−02**	3.18E−02	2.10E−02	−1.11E−02	−1.84E−02	**5.89E−02**	4.51E−02	−3.62E−02	−3.24E−02	−2.21E−02	2.66E−02	**3.41E−02**	−2.62E−02	−4.23E−02	−4.52E−02	3.20E−02	**7.74E−02**	1.32E−02	−9.71E−03	4.23E−02
$RR_{95\%,95\%}$	3.58E−04	**5.87E−04**	5.67E−05	1.58E−06	1.80E−06	4.23E−04	**5.44E−04**	6.22E−05	1.53E−06	1.93E−06	4.28E−04	**5.48E−04**	6.55E−05	1.57E−06	1.88E−06	4.35E−04	**5.00E−04**	6.21E−05	3.35E−06	3.86E−06
$RR_{97\%,97\%}$	4.77E−04	**8.02E−04**	7.67E−05	2.09E−06	2.36E−06	5.83E−04	**7.43E−04**	8.56E−05	2.04E−06	2.58E−06	6.03E−04	**7.60E−04**	9.20E−05	2.07E−06	2.52E−06	6.01E−04	**6.80E−04**	8.43E−05	4.34E−06	5.14E−06
$VaRR_{95\%,95\%}$	1.77E−04	**2.57E−04**	2.71E−05	8.10E−07	9.56E−07	1.94E−04	**2.44E−04**	2.91E−05	7.82E−07	9.64E−07	1.84E−04	**2.36E−04**	2.86E−05	8.14E−07	8.93E−07	1.95E−04	**2.33E−04**	2.97E−05	1.80E−06	1.93E−06
$VaRR_{97\%,97\%}$	2.59E−04	**4.32E−04**	3.89E−05	1.12E−06	1.31E−06	2.82E−04	**3.91E−04**	4.08E−05	1.08E−06	1.40E−06	2.78E−04	**3.85E−04**	4.20E−05	1.13E−06	1.33E−06	2.91E−04	**3.47E−04**	4.23E−05	2.54E−06	2.69E−06
Sensex 30	6−3					6−2					6−1					3−1				
Mean	**4.76E−04**	2.64E−04	1.25E−04	2.00E−04	2.13E−04	2.51E−04	**2.88E−04**	2.34E−04	2.07E−04	2.22E−04	1.45E−04	**3.12E−04**	3.05E−04	2.09E−04	2.20E−04	1.66E−04	2.11E−04	**2.72E−04**	2.13E−04	2.13E−04
Min	−6.63E−02	−8.36E−02	−7.77E−02	−5.31E−02	−5.08E−02	−5.88E−02	−8.36E−02	−7.74E−02	−5.19E−02	−5.21E−02	−5.87E−02	−4.69E−02	−7.08E−02	−5.19E−02	−5.08E−02	−5.87E−02	−5.14E−02	−7.60E−02	−5.11E−02	−5.10E−02
Max	6.88E−02	6.77E−02	7.26E−02	**7.30E−02**	7.44E−02	6.82E−02	6.25E−02	7.26E−02	**7.30E−02**	7.44E−02	6.92E−02	6.25E−02	7.26E−02	7.30E−02	**7.44E−02**	6.92E−02	**9.29E−02**	7.81E−02	7.67E−02	7.54E−02
Std Dev	1.04E−02	9.73E−03	7.72E−03	**6.02E−03**	6.05E−03	1.03E−02	9.64E−03	7.62E−03	**6.01E−03**	6.05E−03	1.02E−02	9.13E−03	7.57E−03	**6.01E−03**	6.03E−03	1.00E−02	9.73E−03	7.61E−03	**6.00E−03**	6.01E−03
Skewness	1.38E−01	2.04E−01	−2.46E−01	1.38E−01	2.21E−01	8.17E−02	−8.24E−02	−7.15E−03	1.28E−01	1.88E−01	1.66E−01	2.25E−01	−2.39E−02	1.42E−01	1.97E−01	1.85E−01	2.85E−01	9.83E−02	2.17E−01	2.10E−01
Kurtosis	4.65E+00	6.65E+00	1.04E+01	1.13E+01	1.14E+01	3.79E+00	6.03E+00	1.11E+01	1.11E+01	1.14E+01	4.13E+00	3.81E+00	1.13E+01	1.11E+01	1.13E+01	4.15E+00	5.01E+00	1.18E+01	1.20E+01	1.16E+01
EMR	**2.77E−04**	6.60E−05	−7.31E−05	1.79E−06	1.47E−05	5.03E−05	**8.72E−05**	3.39E−05	6.15E−06	2.12E−05	−5.55E−05	**1.12E−04**	1.04E−04	8.39E−06	1.90E−05	−4.13E−05	4.13E−06	**6.51E−05**	6.16E−06	6.27E−06
$SR_{Std.Dev}$	**4.58E−02**	2.72E−02	1.62E−02	3.32E−02	3.52E−02	2.44E−02	**2.98E−02**	3.08E−02	3.44E−02	3.67E−02	1.42E−02	3.42E−02	**4.02E−02**	3.48E−02	3.64E−02	1.65E−02	2.17E−02	**3.58E−02**	3.55E−02	3.55E−02
Sortino ratio	**5.32E−02**	1.24E−02	−2.10E−02	3.30E−03	2.56E−02	9.24E−03	**1.56E−02**	1.01E−02	1.16E−02	3.88E−02	−1.01E−02	2.15E−02	**3.15E−02**	1.60E−02	3.53E−02	−7.63E−03	7.79E−04	**1.93E−02**	9.01E−03	9.19E−03
$RR_{95\%,95\%}$	3.32E−04	**3.52E−04**	1.27E−04	3.12E−06	3.57E−06	3.30E−04	**3.68E−04**	1.28E−04	2.94E−06	3.29E−06	**3.45E−04**	3.35E−04	1.27E−04	2.97E−06	3.29E−06	**3.33E−04**	3.25E−04	1.36E−04	4.78E−06	4.69E−06
$RR_{97\%,97\%}$	4.64E−04	**4.93E−04**	1.73E−04	4.44E−06	4.93E−06	4.57E−04	**5.19E−04**	1.74E−04	4.14E−06	4.51E−06	**4.81E−04**	4.66E−04	1.76E−04	4.18E−06	4.53E−06	**4.62E−04**	4.40E−04	1.92E−04	6.64E−06	6.36E−06
$VaRR_{95\%,95\%}$	1.45E−04	**1.50E−04**	5.88E−05	1.34E−06	1.61E−06	1.50E−04	**1.54E−04**	5.88E−05	1.27E−06	1.49E−06	**1.50E−04**	1.45E−04	5.47E−05	1.28E−06	1.47E−06	1.47E−04	**1.50E−04**	5.57E−05	2.08E−06	2.23E−06
$VaRR_{97\%,97\%}$	2.20E−04	**2.37E−04**	8.78E−05	1.94E−06	2.43E−06	2.19E−04	**2.39E−04**	8.75E−05	1.91E−06	2.25E−06	**2.28E−04**	**2.28E−04**	8.35E−05	1.93E−06	2.28E−06	2.23E−04	**2.24E−04**	8.76E−05	3.13E−06	3.21E−06

Table 10
Out-of-sample performance analysis of the proposed EI model using the concatenated series.

	ETDA1	ETDA2	EIM1	EIM2	EIM3	ETDA1	ETDA2	EIM1	EIM2	EIM3	ETDA1	ETDA2	EIM1	EIM2	EIM3	ETDA1	ETDA2	EIM1	EIM2	EIM3
CNX	6–3					6–2					6-1					3-1				
Mean	9.13E−05	**2.99E−04**	1.82E−04	1.96E−04	2.05E−04	1.43E−04	**5.48E−04**	2.64E−04	2.03E−04	2.03E−04	−1.15E−05	3.37E−04	**3.45E−04**	1.93E−04	1.99E−04	1.01E−04	1.46E−04	**3.04E−04**	2.18E−04	2.18E−04
Min	−1.46E−01	−8.33E−02	−6.18E−02	−5.37E−02	−5.17E−02	−9.89E−02	−9.20E−02	−5.20E−02	−5.55E−02	−5.56E−02	−9.89E−02	−5.71E−02	−7.05E−02	−5.42E−02	−5.40E−02	−9.89E−02	−6.45E−02	−7.49E−02	−5.60E−02	−5.17E−02
Max	7.36E−02	6.82E−02	**7.65E−02**	6.97E−02	6.72E−02	7.36E−02	**9.88E−02**	7.65E−02	6.97E−02	6.72E−02	7.36E−02	**7.90E−02**	7.65E−02	6.97E−02	6.72E−02	7.31E−02	7.90E−02	**8.29E−02**	6.96E−02	6.82E−02
Std Dev	1.13E−02	1.06E−02	7.34E−03	**5.93E−03**	5.95E−03	1.12E−02	1.10E−02	6.99E−03	**5.93E−03**	5.98E−03	1.11E−02	1.01E−02	7.50E−03	**5.94E−03**	5.96E−03	1.10E−02	1.04E−02	7.29E−03	**5.86E−03**	5.89E−03
Skewness	−7.96E−01	−9.39E−02	1.82E−01	−2.73E−02	−4.29E−02	−6.78E−02	4.84E−02	4.28E−01	−4.31E−02	−9.17E−02	1.18E−01	1.65E−01	2.06E−02	−4.07E−02	−9.81E−02	1.55E−01	1.56E−01	1.16E−01	−7.24E−02	−8.82E−02
Kurtosis	1.36E+01	5.97E+00	9.29E+00	1.09E+01	1.00E+01	6.20E+00	9.08E+00	8.91E+00	1.11E+01	1.06E+01	6.08E+00	5.10E+00	1.26E+01	1.09E+01	1.03E+01	6.19E+00	5.43E+00	1.30E+01	1.09E+01	1.01E+01
EMR	−1.04E−04	**1.04E−04**	−1.32E−05	7.32E−07	1.00E−05	−5.42E−05	**3.51E−04**	6.65E−05	5.82E−06	5.62E−06	−2.09E−04	1.39E−04	**1.47E−04**	−4.21E−06	1.53E−06	−1.01E−04	−5.62E−05	**1.02E−04**	1.65E−05	1.63E−05
$SR_{Std.Dev}$	8.07E−03	2.83E−02	2.48E−02	**3.31E−02**	3.45E−02	1.28E−02	**5.00E−02**	3.78E−02	3.43E−02	3.40E−02	−1.04E−03	3.32E−02	**4.60E−02**	3.26E−02	3.34E−02	9.20E−03	1.40E−02	**4.17E−02**	3.73E−02	3.71E−02
Sortino ratio	−1.55E−02	**1.71E−02**	−4.08E−03	1.26E−03	1.61E−02	−8.59E−03	**5.70E−02**	2.18E−02	1.01E−02	8.85E−03	−3.23E−02	2.38E−02	**4.58E−02**	−7.49E−03	2.44E−03	−1.58E−02	−9.53E−03	**2.93E−02**	2.30E−02	2.09E−02
$RR_{95\%,95\%}$	4.26E−04	**4.40E−04**	1.18E−04	3.96E−06	4.57E−06	4.16E−04	**4.38E−04**	1.10E−04	3.89E−06	4.66E−06	**4.45E−04**	3.99E−04	1.26E−04	3.70E−06	4.47E−06	**4.41E−04**	3.95E−04	1.38E−04	6.12E−06	7.10E−06
$RR_{97\%,97\%}$	5.97E−04	**6.14E−04**	1.61E−04	5.91E−06	6.62E−06	5.76E−04	**6.08E−04**	1.51E−04	5.74E−06	6.71E−06	**6.17E−04**	5.41E−04	1.77E−04	5.45E−06	6.43E−06	**6.11E−04**	5.42E−04	1.99E−04	8.55E−06	9.72E−06
$VaRR_{95\%,95\%}$	1.92E−04	**1.96E−04**	5.32E−05	1.37E−06	1.77E−06	1.94E−04	**1.99E−04**	4.97E−05	1.37E−06	1.76E−06	**2.00E−04**	1.90E−04	5.29E−05	1.29E−06	1.69E−06	**1.96E−04**	1.81E−04	5.47E−05	2.63E−06	3.20E−06
$VaRR_{97\%,97\%}$	2.75E−04	**3.03E−04**	8.04E−05	2.38E−06	2.90E−06	2.81E−04	**2.91E−04**	7.29E−05	2.46E−06	2.99E−06	**2.93E−04**	2.72E−04	8.18E−05	2.25E−06	2.89E−06	**2.87E−04**	2.62E−04	8.31E−05	4.03E−06	4.90E−06
IBX−50	6−3					6−2					6−1					3−1				
Mean	2.25E−06	1.24E−04	**1.84E−04**	4.03E−06	−8.76E−06	−2.85E−05	2.53E−05	**2.69E−04**	3.03E−05	2.03E−05	1.41E−04	8.21E−05	**2.08E−04**	3.17E−05	2.38E−05	−2.33E−04	**1.80E−04**	9.89E−05	8.83E−06	−7.78E−07
Min	−1.05E−01	−5.42E−02	−4.50E−02	−4.13E−02	−4.07E−02	−9.01E−02	−5.70E−02	−4.50E−02	−4.13E−02	−4.07E−02	−1.03E−01	−8.06E−02	−4.88E−02	−4.18E−02	−4.22E−02	−9.02E−02	−5.77E−02	−5.47E−02	−4.24E−02	−4.20E−02
Max	**7.16E−02**	3.90E−02	4.52E−02	2.47E−02	2.46E−02	**7.16E−02**	4.57E−02	4.76E−02	2.53E−02	2.50E−02	**9.15E−02**	3.73E−02	4.52E−02	2.63E−02	2.64E−02	**9.15E−02**	8.14E−02	5.29E−02	2.73E−02	2.80E−02
Std Dev	1.35E−02	9.57E−03	7.70E−03	**5.88E−03**	5.88E−03	1.27E−02	1.01E−02	7.74E−03	**5.90E−03**	5.91E−03	1.32E−02	9.93E−03	7.72E−03	**5.88E−03**	5.90E−03	1.22E−02	1.10E−02	7.93E−03	**5.90E−03**	5.92E−03
Skewness	−3.68E−01	−1.71E−01	1.33E−01	−2.47E−01	−2.30E−01	2.57E−01	−3.97E−01	1.30E−01	−2.35E−01	−2.19E−01	5.09E−02	−4.59E−01	7.24E−02	−2.32E−01	−2.18E−01	8.66E−02	4.92E−01	3.07E−01	−2.12E−01	−2.15E−01
Kurtosis	6.60E+00	2.13E+00	3.65E+00	2.56E+00	2.44E+00	5.34E+00	2.50E+00	3.25E+00	2.52E+00	2.38E+00	6.87E+00	3.68E+00	3.42E+00	2.55E+00	2.55E+00	5.35E+00	4.65E+00	6.09E+00	2.48E+00	2.49E+00
EMR	−7.18E−05	5.01E−05	**1.10E−04**	−7.00E−05	−8.28E−05	−1.26E−04	−7.27E−05	**1.71E−04**	−6.77E−05	−7.77E−05	4.29E−05	−1.59E−05	**1.10E−04**	−6.63E−05	−7.42E−05	−3.06E−04	**1.07E−04**	2.59E−05	−6.42E−05	−7.38E−05
$SR_{Std.Dev}$	1.67E−04	1.30E−02	**2.40E−02**	6.85E−04	−1.49E−03	−2.24E−03	2.49E−03	**3.47E−02**	5.14E−03	3.43E−03	1.07E−02	8.26E−03	**2.69E−02**	5.39E−03	4.04E−03	−1.92E−02	**1.64E−02**	1.25E−02	1.50E−03	−1.31E−04
Sortino ratio	−8.86E−03	8.30E−03	**3.63E−02**	−1.35E−01	−1.46E−01	−1.72E−02	−1.10E−02	**5.22E−02**	−1.33E−01	−1.42E−01	5.64E−03	−2.43E−03	**3.37E−02**	−1.31E−01	−1.38E−01	−4.18E−02	**1.62E−02**	7.24E−03	−9.75E−02	−1.07E−01
$RR_{95\%,95\%}$	**6.70E−04**	3.67E−04	1.08E−04	2.19E−06	2.61E−06	**6.17E−04**	4.01E−04	1.20E−04	2.19E−06	2.47E−06	**6.63E−04**	4.15E−04	1.20E−04	2.08E−06	2.34E−06	**5.46E−04**	4.76E−04	1.56E−04	3.41E−06	3.97E−06
$RR_{97\%,97\%}$	**9.67E−04**	4.84E−04	1.47E−04	2.97E−06	3.48E−06	**8.86E−04**	5.21E−04	1.62E−04	2.96E−06	3.32E−06	**9.53E−04**	5.50E−04	1.66E−04	2.83E−06	3.17E−06	**7.64E−04**	6.43E−04	2.25E−04	4.49E−06	5.34E−06
$VaRR_{95\%,95\%}$	**2.70E−04**	1.88E−04	4.84E−05	1.00E−06	1.28E−06	**2.52E−04**	2.15E−04	5.94E−05	1.04E−06	1.20E−06	**2.56E−04**	2.10E−04	5.42E−05	9.85E−07	1.16E−06	**2.36E−04**	2.34E−04	6.09E−05	1.77E−06	1.90E−06
$VaRR_{97\%,97\%}$	**4.20E−04**	2.62E−04	7.23E−05	1.50E−06	1.93E−06	**3.84E−04**	2.91E−04	8.15E−05	1.53E−06	1.71E−06	**4.30E−04**	3.01E−04	7.93E−05	1.44E−06	1.61E−06	**3.58E−04**	3.18E−04	9.22E−05	2.36E−06	2.73E−06
Bovespa	6−3					6−2					6−1					3−1				
Mean	−6.98E−05	−6.37E−05	**1.25E−04**	−5.07E−06	−1.51E−05	**3.53E−04**	−8.98E−05	3.34E−04	9.50E−06	−1.12E−05	**3.19E−04**	−4.45E−05	1.42E−04	2.51E−05	−4.33E−06	−5.79E−05	**4.54E−04**	3.05E−04	2.60E−05	2.04E−05
Min	−1.05E−01	−5.41E−02	−4.76E−02	−4.11E−02	−4.23E−02	−8.99E−02	−5.13E−02	−4.76E−02	−4.11E−02	−4.23E−02	−8.99E−02	−8.06E−02	−5.00E−02	−4.08E−02	−4.30E−02	−8.99E−02	−7.90E−02	−5.21E−02	−4.58E−02	−4.20E−02
Max	**9.16E−02**	5.61E−02	4.43E−02	2.58E−02	2.61E−02	**9.16E−02**	5.61E−02	4.89E−02	2.57E−02	2.54E−02	**9.16E−02**	6.58E−02	4.00E−02	2.73E−02	2.70E−02	**9.16E−02**	8.14E−02	5.78E−02	2.70E−02	2.64E−02
Std Dev	1.32E−02	1.11E−02	7.71E−03	**5.92E−03**	5.99E−03	1.35E−02	1.09E−02	7.76E−03	**5.95E−03**	6.03E−03	1.32E−02	1.09E−02	7.74E−03	**5.94E−03**	6.01E−03	1.28E−02	1.19E−02	8.08E−03	5.98E−03	**5.97E−03**
Skewness	−2.48E−01	−1.34E−01	2.13E−01	−2.24E−01	−2.59E−01	4.54E−01	−2.17E−01	1.48E−01	−2.12E−01	−2.51E−01	4.13E−01	−2.85E−01	1.12E−01	−1.97E−01	−2.17E−01	3.94E−01	4.57E−01	4.84E−01	−2.11E−01	−1.79E−01
Kurtosis	5.90E+00	2.23E+00	3.21E+00	2.45E+00	2.70E+00	5.11E+00	1.88E+00	3.25E+00	2.40E+00	2.63E+00	5.92E+00	3.93E+00	3.17E+00	2.37E+00	2.50E+00	5.90E+00	4.87E+00	6.22E+00	2.58E+00	2.19E+00
EMR	−1.02E−04	−9.55E−05	**9.32E−05**	−3.69E−05	−4.69E−05	**2.96E−04**	−1.47E−04	2.76E−04	−4.77E−05	−6.85E−05	**2.62E−04**	−1.02E−04	8.51E−05	−3.22E−05	−6.16E−05	−9.58E−05	**4.16E−04**	2.67E−04	−1.20E−05	−1.76E−05
$SR_{Std.Dev}$	−5.30E−03	−5.74E−03	**1.62E−02**	−8.57E−04	−2.53E−03	2.61E−02	−8.26E−03	**4.30E−02**	1.60E−03	−1.86E−03	**2.42E−02**	−4.08E−03	1.84E−02	4.22E−03	−7.21E−04	−4.51E−03	**3.81E−02**	3.77E−02	4.35E−03	3.41E−03
Sortino ratio	−1.27E−02	−1.37E−02	**2.78E−02**	−4.55E−02	−5.23E−02	3.98E−02	−2.08E−02	**7.87E−02**	−5.99E−02	−7.74E−02	**3.57E−02**	−1.42E−02	2.39E−02	−4.17E−02	−7.27E−02	−1.28E−02	5.93E−02	**6.60E−02**	−1.29E−02	−1.74E−02
$RR_{95\%,95\%}$	**6.29E−04**	4.63E−04	1.28E−04	6.94E−06	8.44E−06	**6.88E−04**	4.37E−04	1.44E−04	6.69E−06	8.06E−06	**6.48E−04**	4.75E−04	1.42E−04	6.52E−06	8.03E−06	**6.36E−04**	5.82E−04	2.14E−04	9.45E−06	1.13E−05
$RR_{97\%,97\%}$	**8.99E−04**	6.17E−04	1.72E−04	9.73E−06	1.26E−05	**9.68E−04**	5.58E−04	1.98E−04	9.59E−06	1.17E−05	**9.06E−04**	6.30E−04	1.95E−04	9.39E−06	1.18E−05	**8.89E−04**	7.90E−04	3.13E−04	1.31E−05	1.53E−05
$VaRR_{95\%,95\%}$	**2.62E−04**	2.35E−04	6.28E−05	2.95E−06	2.99E−06	**2.92E−04**	2.45E−04	6.82E−05	2.68E−06	3.06E−06	**2.71E−04**	2.41E−04	6.46E−05	2.54E−06	3.05E−06	2.68E−04	**2.77E−04**	8.04E−05	4.05E−06	5.32E−06
$VaRR_{97\%,97\%}$	**3.82E−04**	3.28E−04	9.36E−05	4.77E−06	4.93E−06	**4.39E−04**	3.35E−04	9.48E−05	4.24E−06	4.97E−06	**4.29E−04**	3.41E−04	9.54E−05	4.38E−06	4.81E−06	**4.09E−04**	3.96E−04	1.29E−04	6.65E−06	8.18E−06

Table 11
Out-of-sample performance analysis of the proposed EI model using the concatenated series.

Athex	ETDA1	ETDA2	EIM1	EIM2	EIM3	ETDA1	ETDA2	EIM1	EIM2	EIM3
6–3						6–2				
Mean	−7.51E−04	**−9.00E−05**	−1.28E−04	−1.99E−04	−2.06E−04	−7.10E−04	**−9.85E−05**	−1.68E−04	−2.15E−04	−2.07E−04
Min	−1.56E−01	−1.52E−01	−1.02E−01	−7.63E−02	−7.60E−02	−1.55E−01	−1.52E−01	−9.83E−02	−7.51E−02	−7.47E−02
Max	**1.80E−01**	1.16E−01	1.03E−01	6.15E−02	6.29E−02	**1.80E−01**	1.12E−01	1.10E−01	6.02E−02	6.24E−02
Std Dev	2.12E−02	1.49E−02	1.34E−02	**8.46E−03**	8.57E−03	2.05E−02	1.53E−02	1.34E−02	**8.46E−03**	8.54E−03
Skewness	−7.24E−01	−9.32E−01	−2.57E−02	−3.72E−01	−3.39E−01	−2.76E−01	−3.59E−01	1.23E−01	−3.71E−01	−3.35E−01
Kurtosis	1.07E+01	1.68E+01	1.13E+01	7.68E+00	7.32E+00	9.36E+00	1.26E+01	1.11E+01	7.40E+00	7.48E+00
EMR	−6.02E−04	**5.88E−05**	2.08E−05	−5.01E−05	−5.74E−05	−5.62E−04	**5.04E−05**	−1.89E−05	−6.59E−05	−5.86E−05
$SR_{Std.Dev}$	0	0	0	0	0	0	0	0	0	0
Sortino ratio	**1.46E−02**	1.04E−02	7.12E−03	1.15E−03	1.26E−03	**1.35E−02**	1.01E−02	7.05E−03	1.12E−03	1.19E−03
Omega ER	−1.28E−03	−4.08E−04	−3.94E−04	−3.52E−04	−3.61E−04	−1.20E−03	−4.20E−04	−4.35E−04	−3.67E−04	−3.62E−04
$RR_{95\%,95\%}$	**2.29E−03**	1.11E−03	6.04E−04	1.22E−05	1.52E−05	**2.11E−03**	1.18E−03	5.82E−04	1.17E−05	1.44E−05
$RR_{97\%,97\%}$	**3.35E−03**	1.67E−03	9.32E−04	1.80E−05	2.27E−05	**3.07E−03**	1.77E−03	8.84E−04	1.73E−05	2.14E−05
$VaRR_{95\%,95\%}$	**8.24E−04**	3.74E−04	1.88E−04	4.55E−06	5.24E−06	**7.70E−04**	4.06E−04	1.98E−04	4.23E−06	4.93E−06
$VaRR_{97\%,97\%}$	**1.40E−03**	5.98E−04	3.24E−04	7.19E−06	9.17E−06	**1.30E−03**	6.92E−04	3.18E−04	6.93E−06	8.31E−06
6–1						3–1				
Mean	−7.76E−04	2.31E−05	**1.49E−04**	−2.14E−04	−2.25E−04	−8.47E−04	−5.68E−04	**8.84E−05**	−1.97E−04	−2.06E−04
Min	−1.56E−01	−1.44E−01	−9.83E−02	−7.58E−02	−7.51E−02	−1.56E−01	−5.70E−01	−8.77E−02	−7.63E−02	−8.18E−02
Max	**1.80E−01**	1.12E−01	1.09E−01	6.02E−02	6.24E−02	**1.80E−01**	1.11E−01	1.09E−01	6.09E−02	5.58E−02
Std Dev	2.26E−02	1.50E−02	1.25E−02	**8.37E−03**	8.47E−03	2.24E−02	1.85E−02	1.18E−02	**8.33E−03**	8.36E−03
Skewness	−7.06E−01	−2.91E−01	4.92E−02	−3.76E−01	−3.31E−01	−7.30E−01	−8.36E+00	1.60E−01	−4.01E−01	−4.58E−01
Kurtosis	9.17E+00	1.26E+01	1.22E+01	7.30E+00	7.19E+00	9.61E+00	2.49E+02	1.01E+01	7.23E+00	7.67E+00
EMR	−6.27E−04	1.72E−04	**2.98E−04**	−6.55E−05	−7.57E−05	−7.09E−04	−4.29E−04	**2.27E−04**	−5.92E−05	−6.73E−05
$SR_{Std.Dev}$	0	1.54E−03	**1.20E−02**	0	0	0	0	**7.51E−03**	0	0
Sortino ratio	**1.56E−02**	1.01E−02	6.55E−03	1.04E−03	1.15E−03	**1.54E−02**	1.42E−02	6.25E−03	1.19E−03	1.21E−03
Omega ER	−1.35E−03	−2.88E−04	−9.05E−05	−3.65E−04	−3.77E−04	−1.42E−03	−1.05E−03	−1.32E−04	−3.47E−04	−3.55E−04
$RR_{95\%,95\%}$	**2.59E−03**	1.16E−03	5.04E−04	1.02E−05	1.26E−05	**2.51E−03**	1.30E−03	4.78E−04	1.33E−05	1.50E−05
$RR_{97\%,97\%}$	**3.84E−03**	1.75E−03	7.57E−04	1.49E−05	1.87E−05	**3.73E−03**	1.98E−03	7.14E−04	1.83E−05	2.14E−05
$VaRR_{95\%,95\%}$	**9.17E−04**	3.86E−04	1.73E−04	3.90E−06	4.51E−06	**8.74E−04**	4.38E−04	1.71E−04	5.77E−06	6.11E−06
$VaRR_{97\%,97\%}$	**1.55E−03**	6.50E−04	2.88E−04	6.25E−06	7.45E−06	**1.45E−03**	7.05E−04	2.67E−04	8.97E−06	9.59E−06

References

Adams, H., Emerson, T., Kirby, M., Neville, R., Peterson, C., Shipman, P., … Ziegelmeier, L. (2017). Persistence images: A stable vector representation of persistent homology. *The Journal of Machine Learning Research, 18*(1), 218–252.

Ahmed, P., & Nanda, S. (2005). Performance of enhanced index and quantitative equity funds. *Financial Review, 40*(4), 459–479.

Bacon, C. R. (2011). *Practical portfolio performance measurement and attribution*: 568. John Wiley & Sons.

Beasley, J. E., Meade, N., & Chang, T.-J. (2003). An evolutionary heuristic for the index tracking problem. *European Journal of Operational Research, 148*(3), 621–643.

Berwald, J., Gidea, M., & Vejdemo-Johansson, M. (2013). Automatic recognition and tagging of topologically different regimes in dynamical systems. arXiv:1312.2482arXiv preprint.

Blumberg, A. J., Gal, I., Mandell, M. A., & Pancia, M. (2014). Robust statistics, hypothesis testing, and confidence intervals for persistent homology on metric measure spaces. *Foundations of Computational Mathematics, 14*(4), 745–789.

Bruni, R., Cesarone, F., Scozzari, A., & Tardella, F. (2015). A linear risk-return model for enhanced indexation in portfolio optimization. *OR Spectrum, 37*(3), 735–759.

Bruni, R., Cesarone, F., Scozzari, A., & Tardella, F. (2017). On exact and approximate stochastic dominance strategies for portfolio selection. *European Journal of Operational Research, 259*(1), 322–329.

Bubenik, P. (2015). Statistical topological data analysis using persistence landscapes. *The Journal of Machine Learning Research, 16*(1), 77–102.

Bubenik, P. (2018). The persistence landscape and some of its properties. arXiv preprintarXiv:1810.04963

Canakgoz, N. A., & Beasley, J. E. (2009). Mixed-integer programming approaches for index tracking and enhanced indexation. *European Journal of Operational Research, 196*(1), 384–399.

Carlsson, G. (2009). Topology and data. *Bulletin of the American Mathematical Society, 46*(2), 255–308.

Carlsson, G., Ishkhanov, T., De Silva, V., & Zomorodian, A. (2008). On the local behavior of spaces of natural images. *International Journal of Computer Vision, 76*(1), 1–12.

Chazal, F., Fasy, B., Lecci, F., Michel, B., Rinaldo, A., Rinaldo, A., & Wasserman, L. (2017). Robust topological inference: Distance to a measure and kernel distance. *The Journal of Machine Learning Research, 18*(1), 5845–5884.

Chazal, F., Fasy, B., Lecci, F., Michel, B., Rinaldo, A., & Wasserman, L. (2015). Subsampling methods for persistent homology. In *Proceedings of the international conference on machine learning* (pp. 2143–2151).

Chazal, F., Fasy, B. T., Lecci, F., Rinaldo, A., & Wasserman, L. (2014). Stochastic convergence of persistence landscapes and silhouettes. In *Proceedings of the thirtieth annual symposium on computational geometry* (p. 474). ACM.

Chazal, F., Glisse, M., Labruere, C., & Michel, B. (2013). Optimal rates of convergence for persistence diagrams in topological data analysis. arXiv preprintarXiv:1305.6239

Chazal, F., & Michel, B. (2017). An introduction to topological data analysis: fundamental and practical aspects for data scientists. arXiv preprintarXiv:1710.04019

De Silva, V., & Ghrist, R. (2007a). Coverage in sensor networks via persistent homology. *Algebraic & Geometric Topology, 7*(1), 339–358.

De Silva, V., & Ghrist, R. (2007b). Homological sensor networks. *Notices of the American Mathematical Society, 54*(1), 10–17.

DiBartolomeo, D. (2000). *The enhanced index fund as an alternative to indexed equity management*. Boston: Northfield Information Services. https://www.northinfo.com/documents/70.pdf

Edelsbrunner, H., & Harer, J. (2010). *Computational topology: An introduction*. American Mathematical Soc..

Edelsbrunner, H., Letscher, D., & Zomorodian, A. (2000). Topological persistence and simplification. In *Proceedings of the 41st annual symposium on foundations of computer science* (pp. 454–463). IEEE.

Emrani, S., Gentimis, T., & Krim, H. (2014). Persistent homology of delay embeddings and its application to wheeze detection. *IEEE Signal Processing Letters, 21*(4), 459–463.

Fasy, B. T., Lecci, F., Rinaldo, A., Wasserman, L., Balakrishnan, S., Singh, A., & Others (2014). Confidence sets for persistence diagrams. *The Annals of Statistics, 42*(6), 2301–2339.

Filippi, C., Guastaroba, G., & Speranza, M. G. (2016). A heuristic framework for the bi-objective enhanced index tracking problem. *Omega, 65*, 122–137.

Freedman, D., & Chen, C. (2009). Algebraic topology for computer vision. *Computer Vision*, 239–268.

Gatev, E., Goetzmann, W. N., & Rouwenhorst, K. G. (2006). Pairs trading: Performance of a relative-value arbitrage rule. *The Review of Financial Studies, 19*(3), 797–827.

Ghrist, R. (2008). Barcodes: The persistent topology of data. *Bulletin of the American Mathematical Society, 45*(1), 61–75.

Gidea, M. (2017). Topology data analysis of critical transitions in financial networks. Available at SSRN 2903278.

Gidea, M., & Katz, Y. (2018). Topological data analysis of financial time series: Landscapes of crashes. *Physica A: Statistical Mechanics and its Applications, 491*, 820–834.

Goel, A., & Sharma, A. (2019). Deviation measure in second-order stochastic dominance with an application to enhanced indexing. *International Transactions in Operational Research*. doi:10.1111/itor.12629.

Goel, A., Sharma, A., & Mehra, A. (2018a). Index tracking and enhanced indexing using mixed conditional value-at-risk. *Journal of Computational and Applied Mathematics, 335*, 361–380.

Goel, A., Sharma, A., & Mehra, A. (2018b). Index tracking and enhanced indexing using mixed conditional value-at-risk. *Journal of Computational and Applied Mathematics, 335*, 361–380.

Haley, M. R. (2016). Shortfall minimization and the Naive (1/N) portfolio: An out-of-sample comparison. *Applied Economics Letters, 23*(13), 926–929.

Haley, M. R. (2017). K-fold cross validation performance comparisons of six naive portfolio selection rules: How naive can you be and still have successful out-of-sample portfolio performance? *Annals of Finance, 13*(3), 341–353.

Hatcher, A. (2005). *Algebraic topology*. New York: Cambridge University Press.

Heo, G., Gamble, J., & Kim, P. T. (2012). Topological analysis of variance and the maxillary complex. *Journal of the American Statistical Association, 107*(498), 477–492.

Horak, J., Krlín, L., & Raidl, A. (2003). *Deterministicky chaos a jeho fyzikalni aplikace*. Academia.

Jegadeesh, N., & Titman, S. (2001). Profitability of momentum strategies: An evaluation of alternative explanations. *The Journal of Finance, 56*(2), 699–720.

Kasson, P. M., Zomorodian, A., Park, S., Singhal, N., Guibas, L. J., & Pande, V. S. (2007). Persistent voids: A new structural metric for membrane fusion. *Bioinformatics, 23*(14), 1753–1759.

Khasawneh, F. A., & Munch, E. (2014a). Exploring equilibria in stochastic delay differential equations using persistent homology. In *Proceedings of the ASME 2014 international design engineering technical conferences and computers and information in engineering conference*. American Society of Mechanical Engineers. V008T11A034–V008T11A034

Khasawneh, F. A., & Munch, E. (2014b). Stability determination in turning using persistent homology and time series analysis. In *Proceedings of the ASME 2014 international mechanical engineering congress and exposition*. American Society of Mechanical Engineers. V04BT04A038–V04BT04A038

Khasawneh, F. A., & Munch, E. (2016). Chatter detection in turning using persistent homology. *Mechanical Systems and Signal Processing, 70*, 527–541.

Khasawneh, F. A., & Munch, E. (2017). Utilizing topological data analysis for studying signals of time-delay systems. In *Time delay systems* (pp. 93–106). Springer.

Kim, K., Kim, J., & Rinaldo, A. (2019). Time series featurization via topological data analysis. arXiv preprintarXiv:1812.02987v2

Koshizuka, T., Konno, H., & Yamamoto, R. (2009). Index-plus-alpha tracking subject to correlation constraint. *International Journal of Optimization: Theory, Methods and Applications, 1*(2), 215–224.

Kovacev-Nikolic, V., Bubenik, P., Nikolić, D., & Heo, G. (2016). Using persistent homology and dynamical distances to analyze protein binding. *Statistical Applications in Genetics and Molecular Biology, 15*(1), 19–38.

Li, Q., Sun, L., & Bao, L. (2011). Enhanced index tracking based on multi-objective immune algorithm. *Expert Systems with Applications, 38*(5), 6101–6106.

Makridakis, S., Spiliotis, E., & Assimakopoulos, V. (2019). The m4 competition: 100,000 time series and 61 forecasting methods. *International Journal of Forecasting*. doi:10.1016/j.ijforecast.2019.04.014.

Maletić, S., Zhao, Y., & Rajković, M. (2016). Persistent topological features of dynamical systems. *Chaos: An Interdisciplinary Journal of Nonlinear Science, 26*(5), 53105.

Michael, S. (2005). *Applied nonlinear time series analysis: Applications in physics, physiology and finance*: 52. Singapore: World Scientific.

Mileyko, Y., Mukherjee, S., & Harer, J. (2011). Probability measures on the space of persistence diagrams. *Inverse Problems, 27*(12), 124007. doi:10.1088/0266-5611/27/12/124007.

Munch, E. (2017). A user's guide to topological data analysis. *Journal of Learning Analytics, 4*(2), 47–61.

Munch, E., Turner, K., Bendich, P., Mukherjee, S., Mattingly, J., Harer, J., & Others (2015). Probabilistic Fréchet means for time varying persistence diagrams. *Electronic Journal of Statistics, 9*(1), 1173–1204.

Nicolau, M., Levine, A. J., & Carlsson, G. (2011). Topology based data analysis identifies a subgroup of breast cancers with a unique mutational profile and excellent survival. *Proceedings of the National Academy of Sciences, 108*(17), 7265–7270.

Otter, N., Porter, M. A., Tillmann, U., Grindrod, P., & Harrington, H. A. (2017). A roadmap for the computation of persistent homology. *EPJ Data Science, 6*(1), 17.

Packard, N. H., Crutchfield, J. P., Farmer, J. D., & Shaw, R. S. (1980). Geometry from a time series. *Physical Review Letters, 45*(9), 712.

Paulo, W. L. d., de Oliveira, E. M., & do Valle Costa, O. L. (2016). Enhanced index tracking optimal portfolio selection. *Finance Research Letters, 16*, 93–102.

Perea, J. A., Deckard, A., Haase, S. B., & Harer, J. (2015). Sw1pers: Sliding windows and 1-persistence scoring; discovering periodicity in gene expression time series data. *BMC Bioinformatics, 16*(1), 257.

Perea, J. A., & Harer, J. (2015). Sliding windows and persistence: An application of topological methods to signal analysis. *Foundations of Computational Mathematics, 15*(3), 799–838.

Pereira, C. M. M., & de Mello, R. F. (2015). Persistent homology for time series and spatial data clustering. *Expert Systems with Applications, 42*(15–16), 6026–6038.

Phoa, W. (2013). Portfolio concentration and the geometry of co-movement. *The Journal of Portfolio Management, 39*(4), 142–151.

Piangerelli, M., Rucco, M., Tesei, L., & Merelli, E. (2018). Topological classifier for detecting the emergence of epileptic seizures. *BMC Research Notes, 11*(1), 392.

Pun, C. S., Xia, K., & Lee, S. X. (2018). Persistent-homology-based machine learning and its applications – A survey. arXiv preprintarXiv:1811.00252

Ravishanker, N., & Chen, R. (2019). Topological data analysis (TDA) for time series. arXiv preprintarXiv:1909.10604

Reininghaus, J., Huber, S., Bauer, U., & Kwitt, R. (2015). A stable multi-scale kernel for topological machine learning. In *Proceedings of the IEEE conference on computer vision and pattern recognition* (pp. 4741–4748).

Riepe, M. W., & Werner, M. D. (1998). Are enhanced index mutual funds worthy of their name? *The Journal of Investing, 7*(2), 6–15.
Rudolf, M., Wolter, H.-J., & Zimmermann, H. (1999). A linear model for tracking error minimization. *Journal of Banking & Finance, 23*(1), 85–103.
Scozzari, A., Tardella, F., Paterlini, S., & Krink, T. (2013). Exact and heuristic approaches for the index tracking problem with UCITS constraints. *Annals of Operations Research, 205*(1), 235–250.
Seversky, L. M., Davis, S., & Berger, M. (2016). On time-series topological data analysis: New data and opportunities. In *Proceedings of the IEEE conference on computer vision and pattern recognition workshops* (pp. 59–67).
Singh, G., Memoli, F., Ishkhanov, T., Sapiro, G., Carlsson, G., & Ringach, D. L. (2008). Topological analysis of population activity in visual cortex. *Journal of Vision, 8*(8), 11.
Takens, F. (1981). Detecting strange attractors in turbulence. In *Dynamical systems and turbulence, Warwick 1980* (pp. 366–381). Springer.
Taylor, D., Klimm, F., Harrington, H. A., Kramár, M., Mischaikow, K., Porter, M. A., & Mucha, P. J. (2015). Topological data analysis of contagion maps for examining spreading processes on networks. *Nature Communications, 6*(7723). doi:10.1038/ncomms8723.
Truong, P. An exploration of topological properties of high-frequency one-dimensional financial time series data using TDA. Ph.D. dissertationKTH Royal Institute of Technology. http://urn.kb.se/resolve?urn=urn:nbn:se:kth:diva-220355.
Turner, K., Mileyko, Y., Mukherjee, S., & Harer, J. (2014). Fréchet means for distributions of persistence diagrams. *Discrete & Computational Geometry, 52*(1), 44–70.
Umeda, Y., Kaneko, J., & Kikuchi, H. (2019). Topological data analysis and its application to time-series data analysis. *Fujitsu Scientific & Technical Journal, 55*(2), 65–71.
Vandewalle, N., Brisbois, F., Tordoir, X., et al. (2001). Non-random topology of stock markets. *Quantitative Finance, 1*(3), 372–374.
Weng, Y.-C., & Wang, R. (2017). Do enhanced index funds truly have enhanced performance? evidence from the Chinese market. *Emerging Markets Finance and Trade, 53*(4), 819–834.
Zhu, X. (2013). Persistent homology: An introduction and a new text representation for natural language processing. In *Proceedings of the IJCAI* (pp. 1953–1959).
Zomorodian, A., & Carlsson, G. (2005). Computing persistent homology. *Discrete & Computational Geometry, 33*(2), 249–274.