Lemma 0.1.

$$\begin{aligned}
\alpha_i &:= \begin{pmatrix} j & \mapsto & j & \text{if} & 1 \leq j \leq i-1, \\ j & \mapsto & m & \text{if} & j = i \\ j & \mapsto & j-1 & \text{if} & i+1 \leq j \leq m. \end{pmatrix} \\
&= \begin{pmatrix} 1 & ... & i-1 & i & i+1 & ... & m \\ 1 & ... & i-1 & m & i & ... & m-1 \end{pmatrix} = (m\ m-1\ ...\ i+1\ i), \\
\beta_i &:= \begin{pmatrix} j & \mapsto & m-i+j & \text{if} & 1 \leq j \leq i-1, \\ j & \mapsto & j-i+1 & \text{if} & i \leq j \leq m-1, \\ m & \mapsto & m & & \end{pmatrix} \\
&= \begin{pmatrix} 1 & ... & i-1 & i & ... & m-1 & m \\ m-i+1 & ... & m-1 & 1 & ... & m-i & m \end{pmatrix}, \\
\delta_i &:= (\alpha_i)^{-1} \circ \beta_i \circ \alpha_i.
\end{aligned}$$

where

$$\delta_i = \begin{pmatrix} j & \mapsto & m-i+j+\chi(m-i+j,i) & \text{if} & 1 \leq j \leq i-1, \\ j & \mapsto & j & \text{if} & j = i \\ j & \mapsto & j-i+\chi(j-i,i) & \text{if} & i+1 \leq j \leq m. \end{pmatrix}.$$

Proof. Show that α_i is a permutation. We assert that the inverse is

$$\gamma_i = \begin{pmatrix} j & \mapsto & j & \text{if} & 1 \leq j \leq i-1, \\ j & \mapsto & i & \text{if} & j = m \\ j & \mapsto & j+1 & \text{if} & i \leq j \leq m-1. \end{pmatrix}.$$

Then for $1 \leq j \leq i-1$ we have

$$\alpha_i(\gamma_i(j)) = \alpha_i(j) = j$$

and similarly

$$\gamma_i(\alpha_i(j)) = \gamma(j) = j$$

For $j = m$ we have

$$\alpha_i(\gamma_i(m)) = \alpha_i(i) = m$$

,and also

$$\gamma_i(\alpha_i(m)) = \gamma_i(m-1) = m.$$

For $j = i$ we have

$$\alpha_i(\gamma_i(i)) = \alpha_i(i+1) = (i+1-1) = i$$

and

$$\gamma_i(\alpha_i(j) = \gamma_i(\alpha_i(i)) = \gamma_i(m) = i = j$$

For $j \in \{i+1,\ldots,m-1\} \iff j+1 \in \{i+2,\ldots,m\} \implies \{i+1\} \cup \{i+2,\ldots,m\} \implies j+1 \in \{i+1,\ldots,m\}$ we have

$$\alpha_i(\gamma_i(j)) = \alpha_i(j+1) = j+1-i = j$$

.
At last, we have with $j \leq m-1 \implies j \leq m$

$$\gamma_i(\alpha_i(j)) = \gamma_i(j-1).$$

Now $j \in \{i+1,\ldots,m-1\} \implies j-1 \in \{i+1-1,\ldots,m-1-1\} \implies j-1 \in \{i+1-1,\ldots,m-1-1\} \cup \{m-1\} \implies j-1 \in \{i+1-1,\ldots,m-1\}$. Thus

$$\gamma_i(j-1) = j-1+1 = j.$$

Thus we have shown

$$\begin{aligned} j \in \{1\ldots,i-1\} &\implies \gamma_i(\alpha_i(j)) = j \wedge \alpha_i(\gamma_i(j)) = j \\ j = i &\implies \gamma_i(\alpha_i(j)) = j \wedge \alpha_i(\gamma_i(j)) = j \\ j \in \{i+1\ldots,m-1\} &\implies \gamma_i(\alpha_i(j)) = j \wedge \alpha_i(\gamma_i(j)) = j \\ j = m &\implies \gamma_i(\alpha_i(j)) = j \wedge \alpha_i(\gamma_i(j)) = j \end{aligned}$$

$$j \in S \iff j \in \{1\ldots,i-1\} \vee j = i \vee j \in \{i+1,\ldots,m-1\} \vee j = m$$

Therefore

$$j \in S \implies \gamma(\alpha(j)) = j \wedge \alpha(\gamma(j)) = j$$

which shows that α is bijective (one-to-one and onto). Now we assert that the following is the inverse of β_i

$$\varepsilon_i = \left(\begin{array}{lll} j \mapsto i+j-1 & \text{if} & 1 \leq j \leq m-i, \\ j \mapsto i-m+j & \text{if} & m-i+1 \leq j \leq m-1, \\ j \mapsto m & \text{if} & j = m. \end{array} \right).$$

We now distinguish two cases: $i-1 < m-i$ and $i-1 \geq m-i$.
Let us consider first case (I): $i-1 < m-i$.

$$j \in S \iff j \in \{1\ldots,i-1\} \vee j \in \{i,\ldots,m-i\} \vee j \in \{m-i+1,\ldots,m-1\} \vee j = m$$

Let $j \in \{1 \ldots, i-1\}$, we have $j \leq i-1 < m-i \implies j < m-i$ then

$$\beta_i(\varepsilon_i(j)) = \beta_i(i+j-1) = (i+j-1) - i + 1 = j$$

and

$$\varepsilon_i(\beta_i(j)) = \varepsilon_i(m-i+j).$$

Now because of $j \in \{1, \ldots, i-1\}$ we have $m-i+j \in \{m-i+1, \ldots, m-i+i-1\} = \{m-i+1, \ldots, m-1\}$, we can conclude that

$$\varepsilon_i(m-i+j) = i-j+(m-i+j) = j.$$

Thus we have

$$i-1 < m-i \wedge j \in \{0, \ldots, i-1\} \implies \beta_i(\varepsilon_i(j)) = j \wedge \varepsilon_i(\beta_i(j)) = j$$

□

Lemma 0.2. *Let $m \in \mathbb{N}$, $S = \{1, \ldots, m\}$. Then for each $j \in S$ we define a permutation with, where χ is the indicator function.*

$$\chi : \mathbb{R} \times \mathbb{R} \to \{0, 1\} \text{ with } \chi(a, b) = 1 \iff a \geq b.$$

For each $i \in S\delta(i) \in S_m$, where S_m is the symmetric group.

Proof. For now, let us just show $\delta_i : S \to S$. When $1 \leq j \leq i-1$, we can consider two cases:
(C1) $m-i+j \geq i \iff \chi(m-i+j, i) = 1$ and
(C2) $m-i+j < i \iff \chi(m-i+j, i) = 0$.
The $1 \leq j \leq i-1$ and $m-i+j < i$ □