Functions: limits and continuity

MATH 142 – 2018

Calculus 1B

Functions

Calculus is based on the ideas of functions and limits.

Mostly the functions we will deal with in this course will be

- polynomials.
- algebraic functions (involving powers and roots).
- trigonometric functions.
- exponential and logarithmic functions.

These are not all the functions that exist. Consider

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

This is a perfectly valid function, even if you cannot graph it.

Functions – Basic Concepts

Definition

Let D and C be sets.

1. A function $f : D \to C$ is an operation that assigns a unique element of C to each element of D.
2. D is called the *domain* of the function and C is called the *codomain* of the function.
3. The *range* R of the function is the set $\{y \in C \colon f(x) = y \text{ for some } x \in D\}$. Sometimes people write $R = f(D)$.

Mostly we will be dealing with functions $f : D \to \mathbb{R}$ where $D \subset \mathbb{R}$.

- What is the domain of $f(x) = \ln x$?
- What is the range of $f(x) = x^2 - 1$?

Limits

- Limits play an essential role in calculus. Both the derivative and the integral are defined using limits.
- Here is an "intuitive" definition of a limit given by the course text.

  If the values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but not equal to a), then we write

$$\lim_{x \to a} f(x) = L.$$

- Does the following limit exist?

$$\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$$

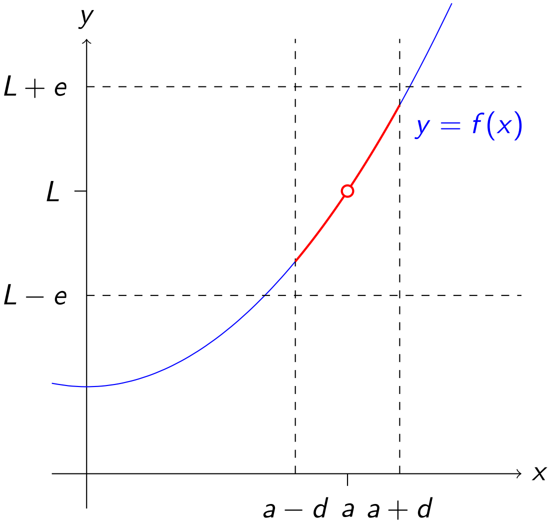
- If $\lim_{x \to a} f(x) = L_1$ and $\lim_{x \to a} g(x) = L_2$ why does

$$\lim_{x \to a} [f(x) + g(x)] = L_1 + L_2?$$

Modern Definition of a Limit

- Developed in the nineteenth century by Bolzano, Cauchy and Weierstrass.
- How do you formalise the idea that $f(x)$ gets close to L as x gets close to a?

An illustration



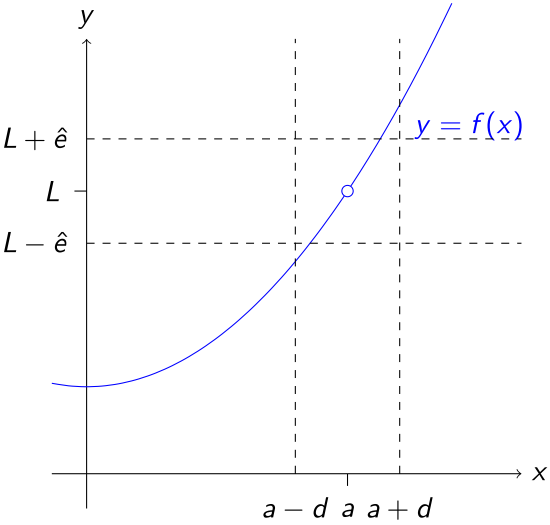
In words

Assume that

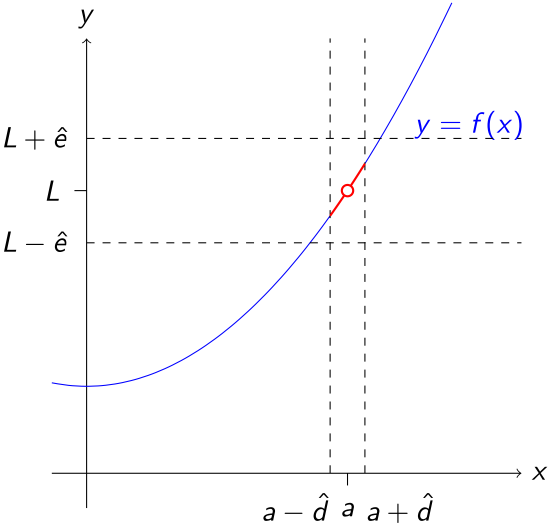
$$\lim_{x \to a} f(x) = L.$$

1. If this is true, then take any positive $e \in \mathbb{R}$ (i.e. $e > 0$) and consider the interval $(L - e, L + e)$.
2. Now imagine x getting closer and closer to a from above. It must be that before reaching a, $f(x)$ gets inside the interval $(L - e, L + e)$ and *stays there*. Similarly as x gets closer and closer to a from below.
3. Hence there is some $d > 0$ such that if $0 < |x - a| < d$ then $|f(x) - L| < e$.

If e changes



Then d can be changed



Definition of a Limit

Definition

Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Let $a, L \in \mathbb{R}$. We say that the limit of $f(x)$ as x tends to a is L, and write

$$\lim_{x \to a} f(x) = L$$

if **for every** $e > 0$, **there is** a $d > 0$ such that
if $0 < |x - a| < d$ then $|f(x) - L| < e$.

This definition is usually written with ε instead of e, and δ instead of d.
It is known as the $\varepsilon - \delta$ limit definition.
Being very formal:

$$\forall \epsilon > 0, \exists \delta > 0 : 0 < |x - a| < \delta \implies |f(x) - L| < \epsilon.$$

(Let's stick to English where possible.)

Example 1.

Definition

Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Let $a, L \in \mathbb{R}$. We say that the limit of $f(x)$ as x tends to a is L, and write

$$\lim_{x \to a} f(x) = L$$

if **for every** $\varepsilon > 0$, **there is** a $\delta > 0$ such that
if $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$.

Show that

$$\lim_{x \to 1} f(x) = 4$$

where f is the function,

$$f(x) = \frac{3x^2 - 2x - 1}{x - 1}.$$

Example 2.

Definition

Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Let $a, L \in \mathbb{R}$. We say that the limit of $f(x)$ as x tends to a is L, and write

$$\lim_{x \to a} f(x) = L$$

if **for every** $\varepsilon > 0$, **there is** a $\delta > 0$ such that
if $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$.

Show that

$$\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$$

does not exist.

Application to limit sums

Definition

Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Let $a, L \in \mathbb{R}$. We say that the limit of $f(x)$ as x tends to a is L, and write

$$\lim_{x \to a} f(x) = L$$

if **for every** $\varepsilon > 0$, **there is** a $\delta > 0$ such that
if $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$.

Show that if $\lim_{x \to a} f(x) = L_1$ and $\lim_{x \to a} g(x) = L_2$ then

$$\lim_{x \to a} [f(x) + g(x)] = L_1 + L_2$$

Properties of limits

Lemma

Suppose that $\lim_{x\to a} f(x) = L_1$, $\lim_{x\to a} g(x) = L_2$ *then*

1. $\lim_{x\to a}[f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L_1 + L_2$.
2. $\lim_{x\to a}[f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) = L_1 - L_2$.
3. $\lim_{x\to a}[f(x)g(x)] = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right) = L_1 \cdot L_2$.
4. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L_1}{L_2}$ *(provided* $L_2 \neq 0$*).*

Other Limits

It is also possible to define rigorously other forms of limit statements such as the following.

1. $\lim\limits_{x \to a^+} f(x) = L$.
2. $\lim\limits_{x \to a^-} f(x) = L$.
3. $\lim\limits_{x \to a} f(x) = +\infty$.
4. $\lim\limits_{x \to a} f(x) = -\infty$.
5. $\lim\limits_{x \to +\infty} f(x) = L$.

We may consider these later in the course if time permits.

When is a function continuous?

Definition

- Given a function f and a number c in its domain, f is *continuous at* $x = c$ if
$$\lim_{x \to c} f(x) = f(c).$$
- A function $f : \mathbb{R} \to \mathbb{R}$ is *continuous* if for every $c \in R$, f is continuous at c.

Continuity

Definition

A function f is *continuous on the interval* (a, b) if it is continuous at every point c in that interval. It is *continuous on the interval* $[a, b]$ if it is continuous on (a, b) and

1. f is continuous from the right at a: $\lim_{x\to a^+} f(x) = f(a)$, and
2. f is continuous from the left at b: $\lim_{x\to b^-} f(x) = f(b)$.

Basic properties

1. Suppose that f and g are continuous at c. Then

   1. $f+g$, $f-g$ and fg are also continuous at c;
   2. f/g is also continuous at c **if** $g(c) \neq 0$.

2. Suppose that f is continuous at c and h is continuous at $f(c)$. Then $h \circ f$ continuous at c. Here $h \circ f$ is the *composition* of h and f:

$$(h \circ f)(x) := h(f(x)).$$

Basic properties

- **Polynomials** e.g. $f(x) = x^5 + 7x^2$ are continuous everywhere.
- **Rational functions:** $f(x) = \dfrac{5x^2 - 7}{x^2 - 1}$ is continuous everywhere **except at** $x = 1$ and $x = -1$ where the denominator $x^2 - 1$ is 0.
- **Algebraic functions:** $\sqrt{x}$ is continuous on $[0, +\infty)$, $\sqrt[3]{x}$ is continuous everywhere.

Basic properties

- **Trigonometric functions:** $\sin x$, $\cos x$ are continuous everywhere; $\tan x$ is continuous **except** when $x = \frac{\pi}{2} + k\pi$ ($k \in \mathbb{Z}$, i.e. an integer).
- **Exponential:** $f(x) = b^x$ (where $b > 0$ is given) is continuous everywhere.
- **Logarithm:** $f(x) = \log_b x$ (where b is given with $b > 0$ and $b \neq 1$) is continuous on $(0, +\infty)$.