1 Finite Differences

We consider 3 different types of finite differences that approximate the discrete version of the derivative:

1. Forward
2. Backward
3. Central

In the following, we define the differences above, in the interior nodes and in the boundary nodes. For notation simplicity, we consider only a **2D matrix and one direction/slice**.

$$u \in \mathbb{R}^{N\times M}, \quad 1 \leq i \leq N, \quad 1 \leq j \leq M$$

DIRECT

1.1 Forward Differences

$$\textbf{Interior}: \quad u_{i+1,j} - u_{i,j}, \quad 2 \leq i \leq N-1, \quad 1 \leq j \leq M$$

Boundary conditions

$$\begin{aligned} \text{Neumann}: \quad u_{2,j} - u_{1,j}, \quad i = 1, \quad 1 \leq j \leq M \\ 0, \quad i = N, \quad 1 \leq j \leq M \end{aligned}$$

$$\begin{aligned} \text{Periodic}: \quad u_{2,j} - u_{1,j}, \quad i = 1, \quad 1 \leq j \leq M \\ u_{N+1,j} - u_{1,j}, \quad i = N, \quad 1 \leq j \leq M \end{aligned}$$

1.2 Backward Differences

$$\textbf{Interior}: \quad u_{i,j} - u_{i-1,j}, \quad 2 \leq i \leq N-1$$

Boundary conditions

$$\begin{aligned} \text{Neumann}: \quad 0, \quad i = 1, \quad 1 \leq j \leq M \\ u_{N,j} - u_{N-1,j}, \quad i = N, \quad 1 \leq j \leq M \end{aligned}$$

$$\begin{aligned} \text{Periodic}: \quad u_{1,j} - u_{N,j}, \quad i = 1, \quad 1 \leq j \leq M \\ u_{N,j} - u_{N-1,j}, \quad i = N, \quad 1 \leq j \leq M \end{aligned}$$

1.3 Centered Differences

$$\textbf{Interior}: \quad \frac{u_{i+1,j} - u_{i-1,j}}{2}, \quad 2 \leq i \leq N-1, \quad 1 \leq j \leq M$$

Boundary conditions

$$\begin{aligned} \text{Neumann}: \quad \frac{u_{2,j} - u_{1,j}}{2}, \quad i = 1, \quad 1 \leq j \leq M \\ \frac{u_{N,j} - u_{N-1,j}}{2}, \quad i = N, \quad 1 \leq j \leq M \end{aligned}$$

$$\begin{aligned} \text{Periodic}: \quad \frac{u_{1,j} - u_{N,j}}{2}, \quad i = 1, \quad 1 \leq j \leq M \\ \frac{u_{N,j} - u_{N-1,j}}{2}, \quad i = N, \quad 1 \leq j \leq M \end{aligned}$$

ADJOINT

We defined the *direct* operation above. Now, in order to define the *adjoint*, we will use the following:

$$< Ax, y > = < x, A^T y > \tag{1.1}$$

where the *LHS* is the direct of the *Finite Difference*. Depending on the boundary conditions, we have the following:

1.1 Forward Differences

$$\textbf{Interior}: \quad u_{i,j} - u_{i-1,j}, \quad 2 \le i \le N-1, \quad 1 \le j \le M$$

$$\begin{aligned} \text{Neumann}: \quad & u_{1,j}, \quad i = 1, \quad 1 \le j \le M \\ & -u_{N-1,j}, \quad i = N, \quad 1 \le j \le M \end{aligned}$$

$$\begin{aligned} \text{Periodic}: \quad & u_{1,j} - u_{N,j}, \quad i = 1, \quad 1 \le j \le M \\ & u_{N,j} - u_{N-1,j}, \quad i = N, \quad 1 \le j \le M \end{aligned}$$

1.2 Backward Differences

$$\textbf{Interior}: \quad u_{i+1,j} - u_{i,j}, \quad 2 \le i \le N-1, \quad 1 \le j \le M$$

$$\begin{aligned} \text{Neumann}: \quad & u_{1,j}, \quad i = 1, \quad 1 \le j \le M \\ & -u_{N,j}, \quad i = N, \quad 1 \le j \le M \end{aligned}$$

$$\begin{aligned} \text{Periodic}: \quad & u_{2,j} - u_{1,j}, \quad i = 1, \quad 1 \le j \le M \\ & u_{1,j} - u_{N-1,j}, \quad i = N, \quad 1 \le j \le M \end{aligned}$$

1.3 Centered Differences

$$\textbf{Interior}: \quad \frac{u_{i+1,j} - u_{i-1,j}}{2}, \quad 2 \le i \le N-1, \quad 1 \le j \le M$$

Boundary conditions

$$\begin{aligned} \text{Neumann}: \quad & \frac{u_{2,j} + u_{1,j}}{2}, \quad i = 1, \quad 1 \le j \le M \\ & \frac{u_{N,j} + u_{N-1,j}}{2}, \quad i = N, \quad 1 \le j \le M \end{aligned}$$

$$\begin{aligned} \text{Periodic}: \quad & \frac{u_{1,j} - u_{N,j}}{2}, \quad i = 1, \quad 1 \le j \le M \\ & \frac{u_{N,j} - u_{N-1,j}}{2}, \quad i = N, \quad 1 \le j \le M \end{aligned}$$

Boundary Conditions

Neumann boundary condition: In this case, we have some **ghost** nodes that are added along the last row/columns of the matrix. So for the case of forward differences, we have the extended matrix below.

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{bmatrix} \longrightarrow \left[\begin{array}{ccc|c} u_{11} & u_{12} & u_{13} & u_{13} \\ u_{21} & u_{22} & u_{23} & u_{23} \\ u_{31} & u_{32} & u_{33} & u_{33} \\ \hdashline u_{31} & u_{32} & u_{33} & \end{array}\right]$$

Periodic boundary condition: In this case, the **ghost** nodes that periodic with respect to the first row/columns of the matrix. So for the case of forward differences, we have the extended matrix below.

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{bmatrix} \longrightarrow \left[\begin{array}{ccc|c} u_{11} & u_{12} & u_{13} & u_{11} \\ u_{21} & u_{22} & u_{23} & u_{21} \\ u_{31} & u_{32} & u_{33} & u_{31} \\ \hdashline u_{11} & u_{12} & u_{13} & \end{array}\right]$$